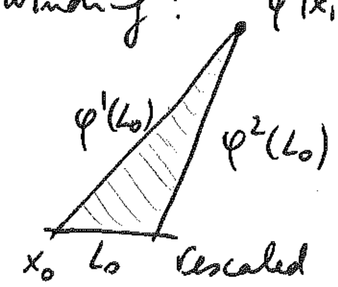
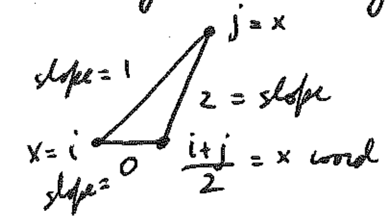
Unwinding : $\varphi^1(x_1)$ $(= x_1$ - fixed pt$)$



$$\boxed{\begin{matrix} CW^*(L_0, L_0) \simeq \mathbb{K}[x^{\pm 1}] \\ x_i \leftrightarrow x^i \end{matrix}} :$$

$$\mu^2(x_1, x_0) = T^{\cdots} x_1$$

- can absorb $T^{\cdots}$ by suitably rescaling generators (by their action)
- $\mu^2(x_i, x_j) = x_{i+j}$



10/20

<u>Wrapped Fukaya category</u>

Liouville mfld : (eg Weinstein mfld of finite type)

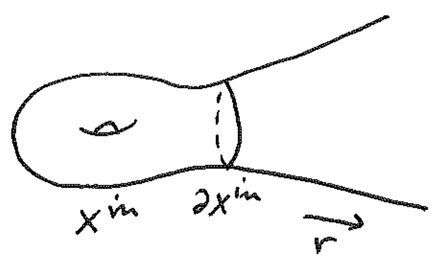
X, $\omega = d\lambda$, Liouville v.f. Z : $\iota_Z \omega = \lambda$ (conformally expanding)

λ: Liouville form; Z: outward pointing at ∞

$X = X^{in} \cup [1,\infty) \times \partial X^{in}$, with r the coordinate on $[1,\infty)$

X^{in}: Liouville domain; ∂X^{in}: contact mfld $(\partial X^{in}, \alpha = \lambda|_{\partial X^{in}})$

$\lambda = r\alpha$, $\omega = dr \wedge \alpha + r\, d\alpha$



<u>Lagrangian submflds</u>:

- exact ($\lambda|_L$ exact) (can weaken)
- conical at ∞ : outside cpct, equals $[1,\infty) \times \Lambda$, Λ = Legendrian submfld of ∂X^{in} : $\alpha|_\Lambda = 0$

  ($\lambda|_L$ cpctly supported)

Wrapped Floer cohomology:

Generators: interior intersection pts + Reeb chords at ∂X^{in} outside cpt set

Quadratic Hamiltonian: $H = \frac{1}{2} r^2 \rightsquigarrow X_H = r \cdot$ Reeb flow of α.

$CW^*(L_0, L_1)$: gen'd by time-1 trajs of X_H, from L_0 to L_1.

Differential: counts $u : \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \to M$ $\quad u(s,0) \in L_0$, $u(s,1) \in L_1$

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H\right) = 0$$

$\longleftrightarrow \quad \tilde{u}(s,t) = \phi_H^{1-t}(u(s,t))$ w/ bdry on $\phi_H^1(L_0)$ & L_1

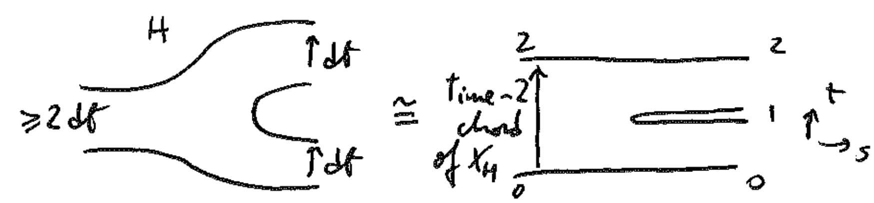
$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}_t\left(\frac{\partial \tilde{u}}{\partial t}\right) = 0 .$$

Product: $CW(L_1, L_2; H) \otimes CW(L_0, L_1; H) \to CW(L_0, L_2; 2H)$

Need β 1-form on domain, st $\beta|_{\text{strip-like ends}} = \underset{\substack{\uparrow \\ \text{constant}}}{k}\, dt$

$\beta|_{\partial \text{domain}} = 0$ (to have Stokes)

Count $(du - X_H \otimes \beta)_J^{0,1} = 0$



eg: $\beta = dt$

Need $\boxed{d\beta \leq 0}$ everywhere in domain, in order to prevent solutions from escaping to $r \to \infty$ $\quad (H \geq 0)$.

Similarly, μ^k wants to land in $CW(L_0, L_k; kH)$

Abouzaid's trick: Liouville flow ψ^k
(time $\log k$ flow of Z) intertwines kH and H
(at ∞, $Z = r\partial_r$, ψ^k multiplies r by k). Get

$$CW(L_0, L_k; H, J) \overset{\text{can}}{\cong} CW(\psi^k(L_0), \psi^k(L_k); k^{-1}(\psi^k)^* H, \psi^k_* J) \quad (*)$$

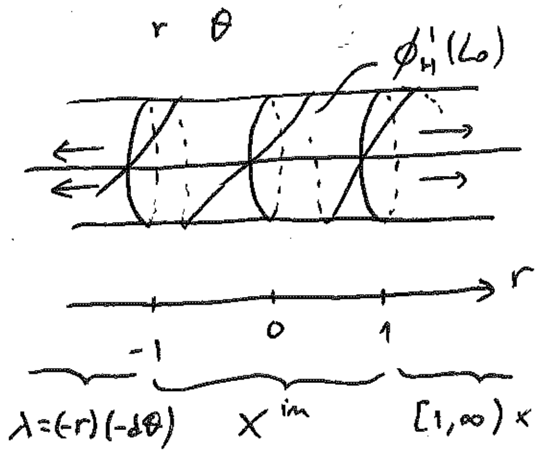
$H = \frac{1}{2} r^2$ at ∞ $\Rightarrow$ $k^{-1}(\psi^k)^* H = \frac{1}{2} k r^2 = kH$ at ∞
$\psi^k(r, x) = (kr, x)$

Observe that the Lagrs & J have changed. Change in J can be dealt w/ by cont. maps.
If the L_i are invt under Liouville flow (even if not, L_i and $\psi^k(L_i)$ are exact Lagr isotopic, by cptly supp isotopy)
$\exists$ continuation maps b/w RHS in $(*)$ & $CW(L_0, L_k; kH, J)$

Naively: compose  w/ continuation $kH \rightsquigarrow H$

In fact: build continuation maps into Floer's equation.

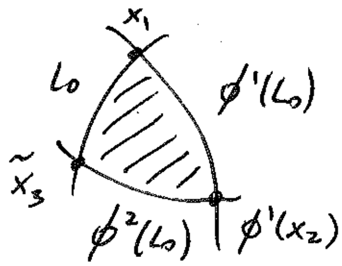
<u>Ex</u>: $X = \mathbb{R} \times S^1$, $\omega = d(\underbrace{r\,d\theta}_{\lambda})$
$r \quad \theta$



$H = \frac{1}{2} r^2$ globally
$\phi^1_H(r, \theta) = (r, \theta + r)$

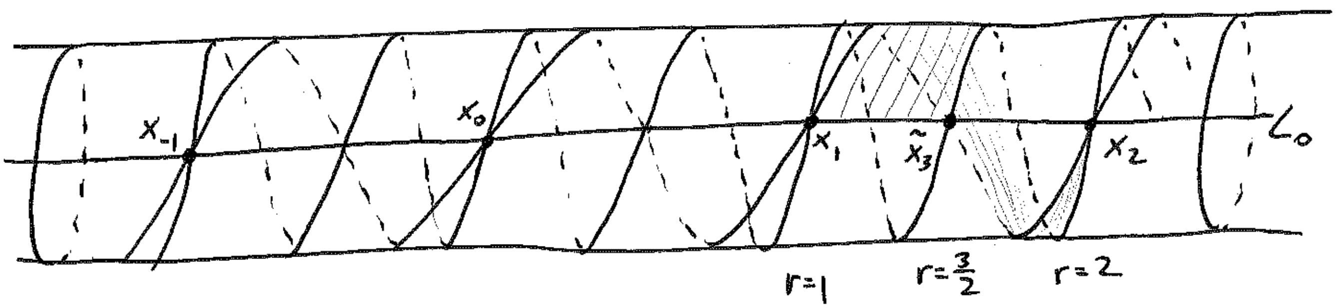
$CW^*(L_0, L_0) = \text{span}\{x_i,\ i \in \mathbb{Z}\}$

$\mu^2 : CW(L_0, L_0) \otimes CW(L_0, L_0) \to CW(L_0, L_0)$



$\tilde{u}(s,t) = \phi^{2-t}(u(s,t))$

L_0 invariant under Liouville flow $\psi^k(r,\theta) = (kr, \theta)$



- $\mu^1 \equiv 0$
- $\mu^2(x_i, x_j) = x_{i+j} \quad \forall i,j \in \mathbb{Z}$ ($\exists\, \mathbb{Z}$-grading / $\deg x_i = 0$)
- $\mu^{\geq 3} = 0$

So, $HW^*(L_0, L_0) \cong \mathbb{K}[x^{\pm 1}]$ ($x_i \leftrightarrow x^i$)

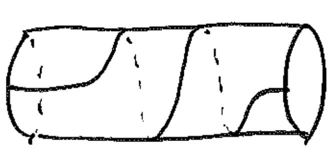
(no higher products)

<u>Abouzaid</u>: L_0 generates $\mathcal{W}(\mathbb{R} \times S^1)$.

$\mathcal{W}($  $) \hookrightarrow \text{mod-}\mathbb{K}[x^{\pm 1}]$

$T \longmapsto CW^*(L_0, T)$

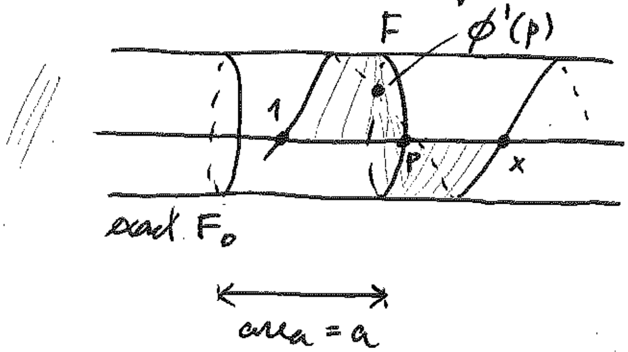
<u>HMS for $\mathbb{R} \times S^1$</u>: $D\mathcal{W}(\mathbb{R} \times S^1) \simeq D^b Coh(\mathbb{K}^*)$

 $\simeq L_0$ (Ham isotopy by a linear Hamilt $\ll H$)

Q: what is $CW^*(L_0, F)$ as a module over $\mathbb{K}[x^{\pm 1}]$?

$$\underset{\mathbb{K}\cdot p}{\overset{\wr\wr}{}} \qquad \underset{CW^*(\phi^1(L_0), F)}{\overset{\wr\wr}{}}$$

F non-exact Lagr w/ unitary local system.



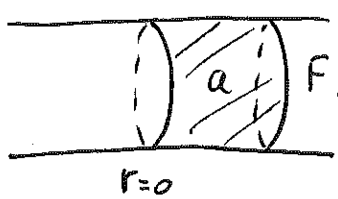
$1 \cdot p = p$ (unitality on H^*)

$x \cdot p = z \cdot p$

$F \leftrightarrow z = T^a, \text{hol} \in \mathbb{K}^*$

Module str: $\mathbb{K}[x^{\pm 1}]/x-z$, ie $\mathcal{O}_{z \in \mathbb{K}^*}$ $\quad L_0 \xrightarrow{x - z\cdot 1} L_0$

F mirror to skyscraper sheaf at pt.

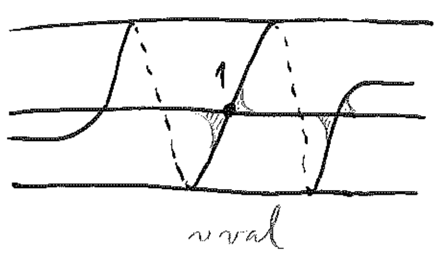


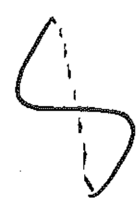
$z = T^a \cdot \underbrace{\text{hol}}_{\in U_{\mathbb{K}}} \in \mathbb{K}^*$

$\in U_{\mathbb{K}}$ unitary elements

$c_0 +$ pos. powers of T

Recall can relate mapping cones w/ surgery:



surgered  $\cong F$

local system related to the details of the surgery

Instead of using quadratic H:

Abouzaid - Seidel considered multiples of a linear Hamiltonian $h = r$ at ∞

wh rotates by w turns.

$$CF(L_{k-1}, L_k; w_k h) \otimes \dots \otimes CF(L_0, L_1; w_1 h) \to CF^*(L_0, L_k; (w_1+\dots+w_k)h)$$

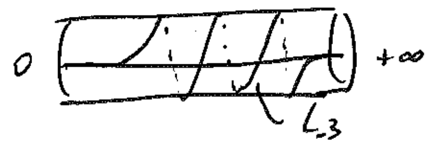
$$CF(L_i, L_j; h) \to CF(L_i, L_j; 2h) \to CF(L_i, L_j; 3h) \to \cdots$$

$$CW^*(L_i, L_j) = \varinjlim_{w \to \infty} CF^*(L_i, L_j; wh) \quad \text{inherits } A_\infty\text{-ops.}$$

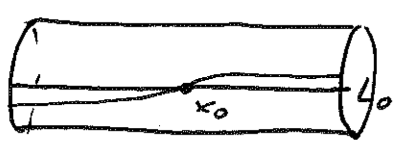
This plays well w/ something we may have seen elsewhere:

Fukaya - Seidel cat of Landau - Ginzburg model $(\mathbb{C}^*, \underbrace{z + \tfrac{1}{z}}_{W})$:

Objs: Lagrs L st $W|_L \to +\infty$ outside cpct set (real positive)



Homs: defined as $CF(L_0, L_1; \varepsilon h)$, $\varepsilon > 0$ small



$\hom(L_0, L_0)$ rank 1

Also have objects L_k, winding around $-k$ times.

$$\hom(L_0, L_k) = \begin{cases} \mathbb{K}^{k+1} & \text{in deg } 0 \quad \text{if } k \geq 0 \\ \mathbb{K}^{|k|-1} & \text{in deg } 1 \quad \text{if } k < 0 \end{cases}$$

$L_k \longleftrightarrow \mathcal{O}(k)$ on $\mathbb{CP}^1$

So,

$$\boxed{\hom_{FS}(L_i, L_j) \leftrightarrow \mathrm{Ext}_{Coh(\mathbb{P}^1)}(\mathcal{O}(i), \mathcal{O}(j))}$$

Observe: $\phi_h^1(L_i) \simeq L_{i+2}$

$$CW^*_{lin}(L_i, L_j) = \varinjlim_{w\to\infty} CF^*(\phi_{wh}(L_i), L_j) = \varinjlim CF^*(L_{i-2w}, L_j) \quad (**)$$

What is this mirror to in Alg. Geom.?

$$\mathbb{C}^* = \mathbb{CP}^1 \setminus \overbrace{\{0,\infty\}}^{D}$$

$$\hom_{\mathbb{C}^*}(\mathcal{O}(i)|_{\mathbb{C}^*}, \mathcal{O}(j)|_{\mathbb{C}^*}) = \varinjlim_{w\to\infty} \hom_{\mathbb{CP}^1}(\mathcal{O}(i-2w), \mathcal{O}(j))$$

connecting map: multipl. by defining eqn of $D = \{0,\infty\}$

This matches the connecting map in $(**)$ above!

So,

$$\begin{array}{c} \mathbb{C}^* \subset \mathbb{CP}^1 \\ \updownarrow \text{HMS} \\ \mathbb{C}^*, W \leftrightarrow (\mathbb{C}^*, z + \frac{1}{z}), \text{FS} \end{array}$$

Can also do partial wrappings (Sylvan).

Eg: if if at 0 and wrap at ∞ in $\mathbb{C}^*$, recover $W(\mathbb{C})$.