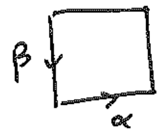
<u>More modern viewpoint</u>: $\mathcal{F}(T^2)$ is split-generated by

2 objects α & β



Formally, every object $\simeq$ direct summand in iterated mapping cone of copies of α, β.

Yoneda embedding (contravariant):

$$\mathcal{C} \longrightarrow \text{mod-}\mathcal{C}$$

$$L \longmapsto \{\underset{\text{chain cxs}}{\hom(T,L)}\}_{T\in\mathcal{C}}$$

+ str. maps (A_∞)

(will take $\mathcal{C} = \mathcal{F}(X)$)

In category of modules, have mapping cones:

given $f: A \to B$ closed ($\mu^1(f) = 0$, $f \in \hom(A,B)$ chain cx)

$$\mathrm{Cone}(A \xrightarrow{f} B)^i = A^{i+1} \oplus B^i = \begin{tikzcd} A[1] \arrow[loop below, "\partial_A"] \arrow[rr, bend left, "f"] & \oplus & B \arrow[loop below, "\partial_B"] \end{tikzcd}$$

(in Alg-Top., this is what chains on mapping cone look like)

Given $A, B \in \mathcal{C}$, $f: A \to B$ closed, say $C \in \mathcal{C}$ is a <u>cone</u> of f if the module assoc. to C is qiso to Cone(f).

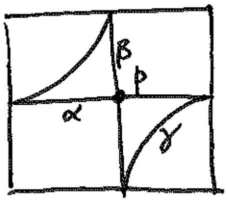
Given a mapping cone as above, have an <u>exact triangle</u>

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd}$$

, hence an associated LES $\forall T$:

$$\cdots \to \hom(T,A) \xrightarrow{f} \hom(T,B) \to \hom(T,C) \to \cdots$$

T^2:



$\mathrm{Cone}(\alpha \xrightarrow{p} \beta) \simeq \gamma$

"Mapping cones are related to surgery".

10/13

Triangles and generators

An exact triangle is

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h", "{[1]}"'] & \end{tikzcd}$$

It induces LES for every T

$$\dots \to H^i \hom(T,A) \xrightarrow{f} H^i \hom(T,B) \xrightarrow{g} H^i \hom(T,C) \xrightarrow{h} \dots$$

these are natural wrt T.

Can always enlarge Fukaya categ. so it has mapping cones.

One way is to take twisted complexes (see Seidel's book)

$Tw(\mathcal{F})$: Obj: • finite collection $\mathbb{E} = \bigoplus_{i=1}^{k} E_i[\sigma_i]$, $E_i \in Obj(\mathcal{F})$

• differential $\delta \in End^1(\mathbb{E})$, ie $\delta_{ij} \in \hom^{\sigma_j - \sigma_i + 1}(E_i, E_j)$

st 1) $\mu^1(\delta) + \mu^2(\delta, \delta) + \dots = 0$

$$\begin{tikzcd} E_1 \arrow[r] \arrow[rr, bend right] & E_2 \arrow[r] & E_3 \end{tikzcd}$$

2) δ strictly triangular: $\delta_{ij} = 0$ unless $i<j$.

($\Rightarrow$ finiteness in 1))

Ex: $E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$ twisted cx if

$\mu^1(f) = 0$, $\mu^1(g) = 0$, $\mu^2(g, f) + \mu^1(h) = 0$ for some $E_1 \xrightarrow{h} E_3$.

<u>Morphisms</u>: hom of twisted complexes = ⊕ homs between summands

$$\begin{tikzcd} E_1 \arrow[r, "\delta_E"] \arrow[d, "f"'] \arrow[dr] & E_2 \arrow[r, "\delta_E"] \arrow[d] \arrow[dl] & \cdots \\ F_1 \arrow[r, "\delta_F"] & F_2 \arrow[r, "\delta_F"] & \cdots \end{tikzcd}$$

$$\mu^1_{Tw}(f) = \sum_{k,l \geq 0} \mu^{k+l+1}(\underbrace{\delta_F, \dots, \delta_F}_{k}, f, \underbrace{\delta_E, \dots, \delta_E}_{l})$$

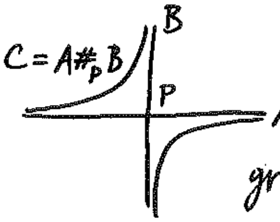
Similarly, given $f_1, \dots, f_k$ k maps between $k+1$ twisted cxs,

$$\mu^k_{Tw}(f_k, \dots, f_1) = \sum \mu(\delta \dots \delta f_k \delta \dots \delta \dots f_1 \delta \dots \delta)$$

These operations satisfy A_∞ relations.

<u>Prop</u>: $Tw(\mathcal{F})$ has mapping cones:
given $(E, \delta_E) \xrightarrow{f} (F, \delta_F)$ st $\mu^1_{Tw}(f) = 0$, then
$\text{Cone}(f) := \left(E[1] \oplus F, \begin{pmatrix} \delta_E & 0 \\ f & \delta_F \end{pmatrix} \right)$.

Mapping cones have at least 2 geometric origins in $\mathcal{F}$:
1) Dehn twists about Lagr spheres [Seidel].
2) Lagr. surgery (connected sum):

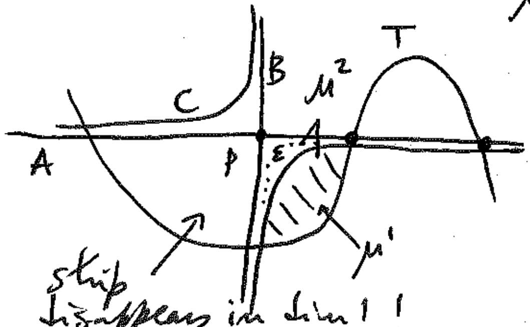


locally graph$(\varepsilon\, d \log \|x\|)$
$\cap$
$T^*\mathbb{R}^n$

$(A \#_p B)$ is a cone of $A \xrightarrow{p} B$
(FOOO, Dehn twist case in dim 1)

- $(B \#_p A) \neq (A \#_p B)$ ( ≠ 

- $$\begin{tikzcd} CF(T,C) \simeq & CF(T,A) \arrow[loop below, "\mu^1"'] \arrow[rr, bend right, "\mu^2(\text{coeff } p, \cdot)"'] & \oplus & CF(T,B) \arrow[loop below, "\mu^1"] \end{tikzcd}$$ in dim > 1:

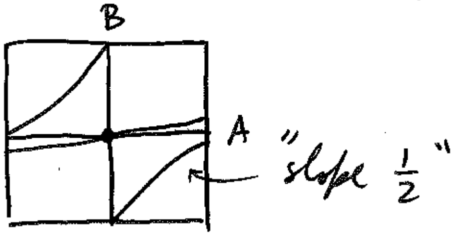


this strip disappears in dim 1!

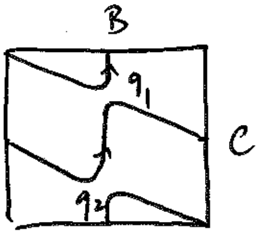
⌞ coeff accounts for $T^{-\varepsilon}$ (area difference b/w  & )
& for difference in local systems

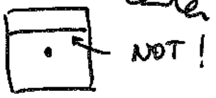
In $Tw(\mathcal{F})$, $C \simeq \{ A \xrightarrow{p} B \}$.

Ex: T^2 : $A, B \in \mathcal{F}(T^2)$



In smallest full subcat of $T\mathcal{F}(T^2)$ containing A & B & mapping cones, can build curves representing any slope on T^2, but only balanced wrt 180° rotation about center





$$\mathrm{Cone}\left(C \xrightarrow{T^{q_1}q_1 + T^{q_2}q_2} B\right) \simeq A_1 \oplus A_2$$

both isotopic (non-Ham) to A.

Every obj of $\mathcal{F}(T^2)$ is $\simeq$ to a direct summand in a twisted complex built from A, B:
"A, B split-generate $\mathcal{F}(T^2)$".

Note: Coh is split-closed, but $\mathcal{F}$ is not in general.

Eg: $T_{cl} \subset \mathbb{P}^2$ splits as two summands that are not geometric (at least in char 0).

So, need to take split-closure of $\mathcal{F}$.

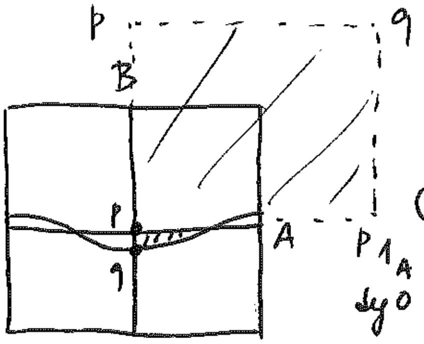
"To compute $\mathcal{F}(T^2)$, enough to compute for A, B, w/ all A_∞-structure".

Have functor

$$\mathcal{F}(T^2) \hookrightarrow \text{mod-}\mathcal{A} \quad A_\infty\text{-modules}$$

$$T \longmapsto CF(T,A) \oplus CF(T,B)$$

$\mathcal{A} = \mathrm{End}(A \oplus B)$.



$$\begin{tikzcd} A \arrow[r, shift left, "\overset{\deg 0}{p}"] \arrow[loop below] & B \arrow[l, shift left, "\underset{\deg 1}{q}"] \arrow[loop below] \end{tikzcd}$$

P $1_A, f_A$
deg0 deg1

$1_B, f_B$
deg0, deg1

$\mu^1 \equiv 0$
$\mu^2(p,q) = f_B$
$\mu^2(q,p) = f_A$

$\mu^3(p,q,p) = 0\,p$

L

cancels out

Lekili - Perutz : non-trivial μ^6, μ^8.

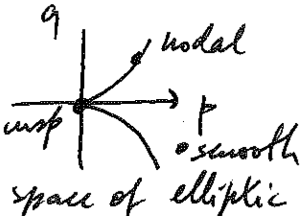
<u>Goal</u>: find small collection of generating objects, for which one doesn't have to compute very high μ^k.

<u>Lekili – Perutz</u>: A_∞ - strs on A are classified by two scalars (μ^6, μ^8)

$\updownarrow$

(not same p, q as above!)

mirror elliptic curve : $y^2 = x^3 + \boxed{p}x + \boxed{q}$
Weierstrass form up to rescaling action

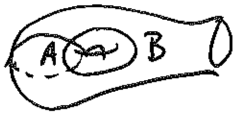


space of elliptic curves

area (T^2) ⟷ modular parameter of mirror

$\int_{T^2}\omega = \alpha$, B-field... $\Lambda^*/z \sim T^\alpha z$

When puncture T^2, might expect to not see holo curves, but still have $\mu^6, \mu^8 \not\equiv 0$! When take Ham perturbs, there are hexagons & octogons

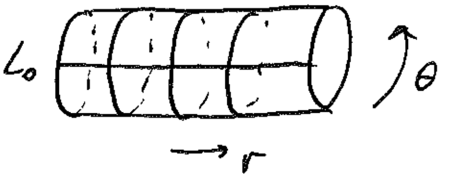


⟷

nodal elliptic curve

The calculation is done via HH_*.

Ex: $\mathbb{R} \times S^1$:



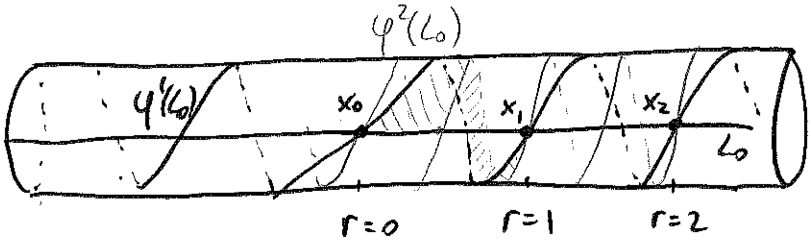
$\longleftrightarrow \quad \mathrm{Coh}(\mathbb{K}^*) = \text{f.g mod-}\mathbb{K}[x^{\pm 1}]$

↑ not cptly supp, if allow non-closed Lagrs on left

Wrapped Fukaya category of $X = \mathbb{R} \times S^1$: (Abouzaid - Seidel)

Floer thry w/ Ham perturbation growing quadratically at ∞ : $H = \frac{1}{2} r^2$.

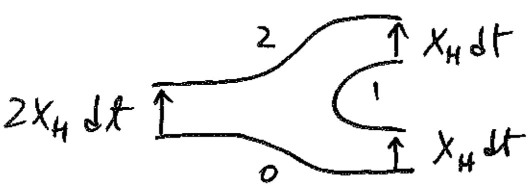
$\omega = dr \wedge d\theta$, $\quad X_H = r\, d\theta$

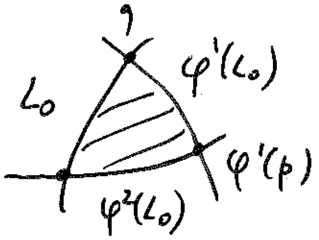


$$CW^*(L_0, L_0) = \bigoplus_{n \in \mathbb{Z}} \mathbb{K}\, x_n \quad , \quad \text{all in deg } 0$$

$\Rightarrow$ diff $= 0$.

Product: $\underset{p}{CF^*(L_0, L_0; H)} \otimes \underset{q}{CF^*(L_0, L_0; H)} \to CF^*(L_0, L_0; 2H)$





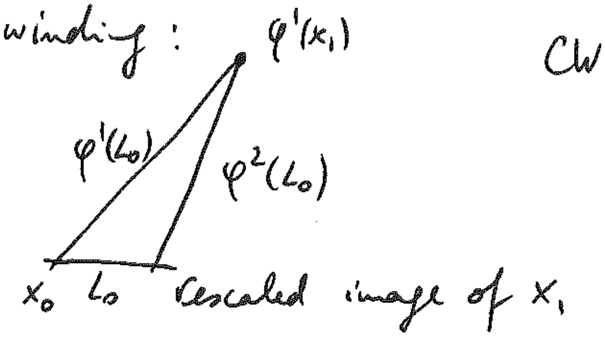
For well-definedness, need H to grow from input to output

$$\tilde{u}(s,t) = \varphi_H^{2-t}(u(s,t))$$

time-1 chords $L_0 \to L_0$ $\quad \xleftrightarrow[\text{by } r \to 2r \text{ Liouville flow}]{\text{intertwined}}$ $\quad$ time-2 chords

$\cong \varphi^1(L_0) \cap L_0 \qquad\qquad \cong \varphi^2(L_0) \cap L_0$

Unwinding :



$$CW^*(L_0, L_0) \simeq \mathbb{K}[x^{\pm 1}]$$
$$x_i \leftrightarrow x^i$$

$$\mu^2(x_1, x_0) = T^{\cdots} x_1$$

- can absorb $T^{\cdots}$ by suitably rescaling generators (by it's action)
- $\mu^2(x_i, x_j) = x_{i+j}$