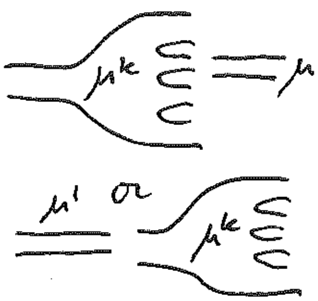
If $\mathcal{M}(p_1,\dots,p_k,q,[u],J)$ is 1-dim,

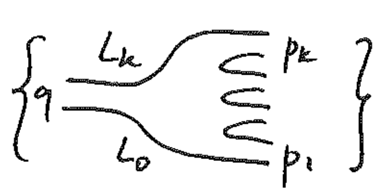
then generically have 1-param family of domains:

Codim 1 faces = pairs of disks
(generically, avoid higher codim faces)

So, the boundary of a 1-dim $\mathcal{M}(p_1,\dots,p_k,q,[u],J)$ is

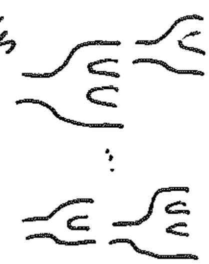


strip breaking $\longleftarrow$ (energy escapes at one of the ends)



(assuming no disk bubbling)

domain degenerates $\longrightarrow$ to union of two disks



(Trace energy escaping and breaking of domain).
terms w/ μ^1 — terms w/o μ^1

<u>Note</u>: Naively, $\exists$ cyclic symmetry $CF^*(L_0,L_k) \simeq CF^{n-*}(L_0,L_k)^\vee$.

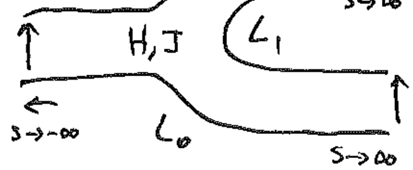
For $\langle \mu^k(p_k,\dots,p_1), q\rangle$, $\exists$ cyclic symmetry, <u>up to homotopy</u>.

$\rightsquigarrow$ Fukaya categories are <u>Calabi-Yau</u> A_∞-categories.

<u>Hamiltonian perturbations</u>

We defined $(CF^*(L_0,L_1),\partial)$ using H_{L_0,L_1} & J_{L_0,L_1}.

For product:



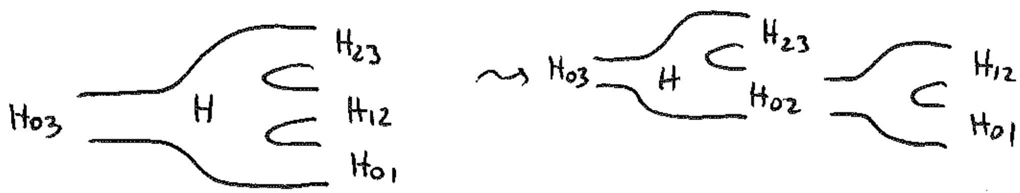
Choose strip-like ends : charts $z = s+it$ near punctures.

Eq: $\frac{\partial u}{\partial s} + J^t \left(\frac{\partial u}{\partial t} - X_{H^t} \right) = 0$ near ends.

This is the Floer eq w/ correct perturbations near ends.

That's it for strip-breaking.

What about <u>domain-breaking</u>?



H must vary differently w/ different domains!

<u>Thm</u> (Seidel): $\exists$ inductive procedure for constructing consistent families of (H,J).

The set of choices at each step is contractible (see Seidel's book).

This uses ~~the~~ fact that associahedron is contractible.

10/6

<u>Fukaya category</u>

(M,ω) cpct or w/ reasonable behavior at infinity.

Objs = compact Lagrangian submflds, unobstructed (no holom discs), + spin structures (for char $\neq 2$), (+ gradings if $2c_1(M)=0$ and want $\mathbb{Z}$-grading), + local systems ("unitary" flat bundle $/L$).

We'll consider rank 1 local systems, on trivial bundles over $\mathbb{K} = \Lambda$ Novikov field.

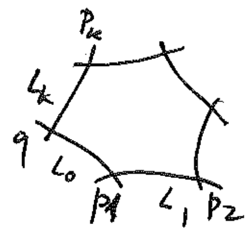
Holonomy $\in \mathrm{Hom}(\pi_1(L), U_\Lambda) = H^1(L; U_\Lambda)$,

$U_\Lambda = \{ a_0 + \sum_{\lambda_i > 0} a_i T^{\lambda_i} \mid a_0 \neq 0 \}$.

Morphisms: $CF((L, \mathcal{E}), (L', \mathcal{E}')) = \bigoplus_{p \in \mathcal{X}(L,L')} \hom(\mathcal{E}_p, \mathcal{E}'_p)$

Operations: $\mu^k(x_k, -, x_1) = \sum_{\substack{q \in \mathcal{X}(L_0, L_k) \\ [u] \mid \mathrm{ind} = 2-k}} (\#\mathcal{M}) \, T^{\omega([u])} \, \mathrm{hol}(x_k, -, x_1)$

where $\mathrm{hol}(x_k, -, x_1)$ = product of parallel transp of $\mathcal{E}_i$'s along bdry on L_i and inputs $x_i \in \hom((\mathcal{E}_i)_{p_i}, (\mathcal{E}_{i+1})_{p_{i+1}})$



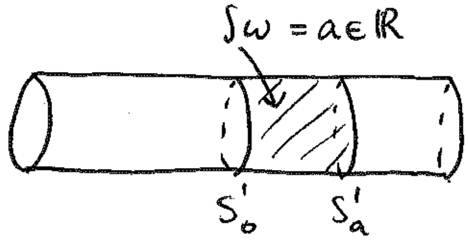
<u>Note</u>: Modified weight of $[u]$ from $T^{\omega([u])}$ to $T^{\omega([u])} \cdot \mathrm{hol}([\partial u])$.

We fix choices H, J for <u>all pairs</u> of objects & consistent Floer data.

Different choices $\rightsquigarrow$ quasi-isomorphisms, via continuation maps.

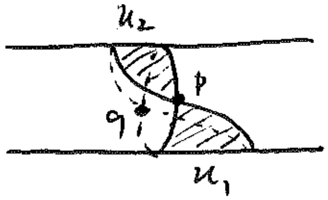
<u>Ex</u>: cylinder: $X = T^*S^1 = \mathbb{C}^* = \mathbb{R}_r \times S^1_\theta$

Don't allow contractible loops, because bound discs (would be zero object anyway).



these are all objects, up to Ham isotopy. $\mathrm{hol} = \xi \in U_\Lambda$.

$$HF^*\big((S^1_a, \xi), (S^1_{a'}, \xi')\big) \simeq \begin{cases} H^*(S^1; \Lambda) & , \text{ if } (a,\xi) = (a', \xi') \\ 0 & , \text{ o/w} \end{cases}$$



$$(*) \quad \mu^1(p) = \Big(T^{\omega(u_1)} \mathrm{hol}(\partial u_1) \underset{\uparrow}{-} T^{\omega(u_2)} \mathrm{hol}(\partial u_2)\Big) q.$$

if use same spin str. on all objects

$$\frac{T^{\omega(u_2)} \mathrm{hol}(\partial u_2)}{T^{\omega(u_1)} \mathrm{hol}(\partial u_1)} = T^{\omega(u_2 - u_1)} \mathrm{hol}\big(\partial(u_2 - u_1)\big) =$$
$$= T^{a'-a}\, \xi'/\xi = 1 \text{ iff } (a,\xi) = (a', \xi')$$

Since different objects don't talk to each other, only need study A_∞ algebra of each object.
Can check: products $\overset{A_\infty}{\cong} H^*(S^1, \Lambda)$.

<u>Other viewpoint</u>: exact Fukaya category : $\omega = d(\underbrace{r\,d\theta}_{\lambda})$

L <u>exact</u> if $\lambda|_L$ is exact. $\int \lambda = a$. Have only S^1_0.
Can work over any coeffs and allow any local systs, not only unitary,

$$(S^1_0, \mathrm{hol} = T^a \underset{\substack{\uparrow \\ \in U_\Lambda}}{\xi}) \longleftrightarrow (S^1_a, \mathrm{hol} = \xi) \qquad \in H^1(S^1_0, \Lambda^*)$$

The two categories are equivalent!
But don't know of a larger category where both left & right sides would live and be isomorphic, because they are disjoint (there's an ad-hoc solution in this case, using flux).

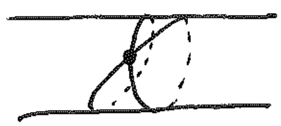
Can take direct sums of objects. Also,

$\exists$ <u>non-trivial extension</u> of (S^1_a, ξ) by itself

$\longleftrightarrow$  S^1_a, rank 2 local syst, hol $= \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}$

S^1_a Jordan form

Could also allow <u>immersed Lagrs</u>:



$$CF^* = \left(\quad C^*(S^1) \oplus \underbrace{\Lambda p \oplus \Lambda q}_{\text{small Ham perturb:}} \right)$$

Can check: ~~[illegible]~~ same as

$\left(S^1, \text{rk } 2 \text{ local syst}, \text{ hol } \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right)$

$\simeq \oplus\, 2$ objects (w/ opposite holonomies)

"This is a uniquely 1-dim phenomenon".

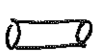
($\mathcal{F}_{cpct, exact}(T^*N) \simeq$ local systems on N – Nadler-Zaslow
only unobstructed objs if $w_2(N) = 0$)

(Higher rank local systems decompose into rank 1 l.s. if $L = T^n$).

Can match

$(S^1_a, \xi) \longleftrightarrow \mathbb{K}[x^{\pm 1}] / (x - T^a \xi)$ $\dim_{\mathbb{K}} = 1$ vector space, torsion module $/\mathbb{K}[x^{\pm 1}]$

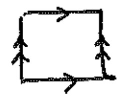
$\longleftrightarrow$ skyscraper $\mathcal{O}_p$, $p = T^a \xi \in \mathbb{K}^* = X^\vee$ mirror

$\mathcal{F}($  $) \simeq \mathrm{Coh}_{cpct}(X^\vee)$

If calculate

$\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_{p'})$

by using resolution $\{ \mathcal{O} \xrightarrow{x - T^a \xi} \mathcal{O} \}$, the differential matches μ^1 $(*)$ in previous page. Family Floer theory: do this more generally.

Ex: $X = T^2$ w/ total area $= A$. 

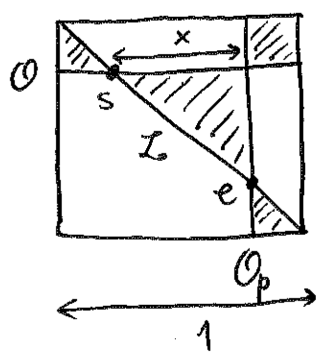
$\forall$ primitive classes in $\pi_1(T^2) = \mathbb{Z}^2$,
$\exists$ family of objects (γ scc, $\xi \in U_\Lambda$).

$(T^a \xi) \in \mathbb{K}^* / (z \sim T^A z) = X^\vee$ "elliptic curve"

At least over $\mathbb{C}$, $\underbrace{\mathrm{Jac}}_{\text{parametrizing line bdles}}(X^\vee) = X^\vee$.

Think of loops in class (p,q) as $\underbrace{\text{rank } p, \text{ deg } q}_{\text{generic bdles (semi-stable ?...) are determined by } p \,\&\, q}$, $\gcd(p,q)=1$, vector bdles.
$(0,1)$: skyscraper sheaves of points.

Can calculate:



$\mathcal{L}$ deg 1 line bdle

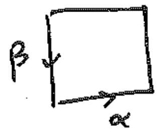
$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"] \arrow[rr, bend right, "s(p)"'] & \mathcal{L} \arrow[r, "ev_p"] & \mathcal{O}_p \end{tikzcd}$$

$s(p)$ in some triv of $\mathcal{L}_p$

Get series
$$\sum_{n \in \mathbb{Z}} T^{\frac{(n+x)^2}{2} A} \, \mathrm{hol}^{n+x}$$

Can compare w/ theta functions. Discrepancies come from choices of trivializations.

For more, see Polishchuk-Zaslow & Polishchuk.

<u>More modern viewpoint</u>: $\mathcal{F}(T^2)$ is split-generated by 2 objects α & β 

.

Formally, every object $\simeq$ direct summand in iterated mapping cone of copies of α, β.

Yoneda embedding (contravariant):

$$\begin{aligned} \mathcal{C} &\longrightarrow \text{mod-}\mathcal{C} \\ L &\longmapsto \{\underset{\text{chain cxs}}{\hom(T, L)}\}_{T \in \mathcal{C}} \\ &\qquad + \text{str. maps } (A_\infty) \end{aligned}$$

(will take $\mathcal{C} = \mathcal{F}(X)$)

In category of modules, have mapping cones:
given $f: A \to B$ closed ($\mu^1(f) = 0$, $f \in \hom(A,B)$ chain cx)

$$\mathrm{Cone}(A \xrightarrow{f} B)^i = A^{i+1} \oplus B^i = \underset{\circlearrowleft\, \partial_A}{A[1]} \xrightarrow{f} \underset{\circlearrowleft\, \partial_B}{B}$$

(in Alg-Top., this is what chains on mapping cone look like)

Given $A, B \in \mathcal{C}$, $f: A \to B$ closed, say $C \in \mathcal{C}$ is a <u>cone</u> of f if the module assoc. to C is qiso to $\mathrm{Cone}(f)$.

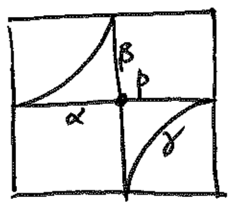
Given a mapping cone (as above), have an <u>exact triangle</u>

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd}$$

, hence an associated LES $\forall T$:

$$\dots \to \hom(T, A) \xrightarrow{f} \hom(T, B) \to \hom(T, C) \to \dots$$

T^2:



$\mathrm{Cone}(\alpha \xrightarrow{p} \beta) \simeq \gamma$

"Mapping cones are related to surgery".

10/13

<u>Triangles and generators</u>

An <u>exact triangle</u> is
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[dl, "g"] \\ C \arrow[u, "h", "{[1]}"'] & \end{tikzcd}$$

It induces LES for every T

$$\dots \to H^i \hom(T,A) \xrightarrow{f} H^i \hom(T,B) \xrightarrow{g} H^i \hom(T,C) \xrightarrow{h} \dots$$

These are natural wrt T.

Can always enlarge Fukaya categ. so it has mapping cones.

One way is to take <u>twisted complexes</u> (see Seidel's book)

$Tw(\mathcal{F})$: <u>Obj</u>:
- finite collection $\mathbb{E} = \bigoplus_{i=1}^{k} E_i[\sigma_i]$, $E_i \in Obj(\mathcal{F})$
- differential $\delta \in End^1(\mathbb{E})$, ie $\delta_{ij} \in \hom^{\sigma_j - \sigma_i + 1}(E_i, E_j)$

st 1) $\mu^1(\delta) + \mu^2(\delta,\delta) + \dots = 0$

$E_1 \to E_2 \to E_3$

2) δ strictly triangular: $\delta_{ij} = 0$ unless $i<j$.
($\Rightarrow$ finiteness in 1))

<u>Ex</u>: $E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$ twisted cx if

$\mu^1(f) = 0$, $\mu^1(g) = 0$, $\mu^2(g,f) + \mu^1(h) = 0$ for some $E_1 \xrightarrow{h} E_3$.