$HF^*(L, L)$?

Local case: L = zero section $\subset T^*L$.

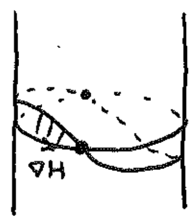
Pick $\varepsilon H : L \to \mathbb{R}$ Morse function (lift to $T^*L \to \mathbb{R}$)

$\phi^1_{\varepsilon H}(L) = \text{graph}(\varepsilon\, dH)$

$\mathcal{X}(L, L, \varepsilon H) = \text{Crit}(H)$

Clever choice of J: along zero-section, want

$J(\nabla H) = \pm X_H$ (want identify TL w/ T^*L fibers, which is same as choosing a metric)



Morse trajs on L $\quad \overset{u(s,t) = (\gamma(s), 0)}{\underset{\varepsilon \to 0}{\longleftrightarrow}} \quad \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_{\varepsilon H}\right) = 0$

$\dot{\gamma}(s) = -\varepsilon \nabla H(\gamma(s))$

<u>Floer's Thm:</u> $HF^*(L, L) \cong HMorse^*(H) = H^*(L)$ in T^*L.

Case of general (M, ω): understand Floer sols as $\varepsilon \to 0$

(Floer, Fukaya-Oh, Biran-Cornea, ...).

As $\varepsilon \to 0$, strips look like

thin part, closer and closer to a grad flow line of H

Solutions converge to union of
$\begin{cases} \text{gradient flow lines of } H \\ J\text{-holo disks w/ boundary on } L \end{cases}$

If no disks (eg $\omega \cdot \pi_2(M, L) = 0$), then $HF^*(L, L) \simeq H^*(L)$.

If there <u>are</u> disks, might not even have $\partial^2 = 0$!

If L <u>monotone</u> ($\mu(\text{disks}) = K\, \omega(\text{disks})$ for some $K > 0$)

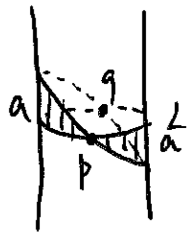
OR $\begin{cases} \text{no disks of } \mu < 2 \text{ (except consts)}, \\ \text{AND } \mu = 2 \text{ disks are regular} \end{cases}$ then $\boxed{\partial^2 = 0}$ on $CF^*(L, L)$.

In such cases, can <u>filter</u> CF^* by Maslov index.

Get a <u>spectral sequence (Oh)</u> $H^*(L) \Rightarrow HF^*(L,L)$

(not $\mathbb{Z}$-graded, if non-trivial disks)

<u>Ex</u>: •



in T^*S^1

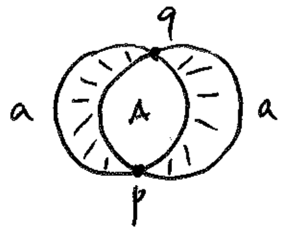
$\partial p = (T^a - T^a)\, q = 0$

$\partial q = 0$

in Morse theory:

$HF^*(L,L) = H^*(L)$

•



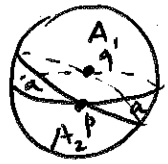
in $\mathbb{C}$

$\partial p = (T^a - T^a)\, q = 0$

$\partial q = T^A p$

$HF^* = 0$

•



equator

in $\mathbb{CP}^1$

$\partial p = (T^a - T^a)\, q = 0$

$\partial q = (T^{A_1} - T^{A_2})\, p = 0$

$HF^*(L,L) = H^*(S^1)$ if $A_1 = A_2$

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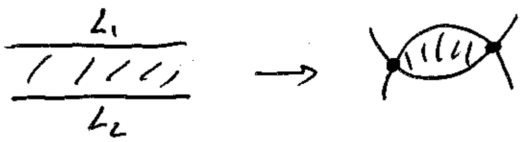
<u>Products and Fukaya category</u>

$L_0, L_1 \rightsquigarrow CF^*(L_0, L_1) = \bigoplus_{p \in \mathcal{X}(L_0,L_1)} \Lambda \cdot p$

$\circlearrowleft$

∂ counts index-1 (perturbed) J-holo maps

$\mathbb{R} \times [0,1]$

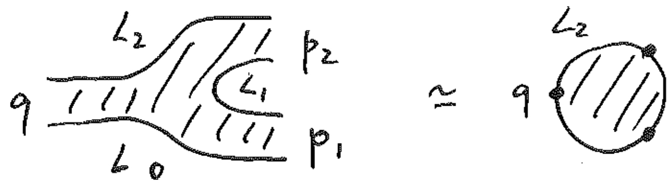


Similarly,

$L_0, L_1, L_2 \rightsquigarrow CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$

(will be composition in Fukaya category)

Given $p_1 \in \mathcal{X}(L_0, L_1)$, $p_2 \in \mathcal{X}(L_1, L_2)$, $q \in \mathcal{X}(L_0, L_2)$, have



(no non-trivial automorphisms of this domain)

$\mathcal{M}(p_1, p_2, q; [u], J) = \{ u : D \to M$ bdry arcs $\mapsto L_i$ $\mid \bar\partial_J u = 0$, $[u]$ given, $u \to p_1, p_2, q$ at bdry punctures $\}$

Expected dim = index of linearized op. = $\mathrm{ind}([u])$ =

$= \deg(q) - \deg(p_1) - \deg(p_2)$

when $\exists$ $\mathbb{Z}$-grading ($2c_1(TM) = 0$, Maslov classes $(L_i) = 0$)

If linearized op. is onto, then $\mathcal{M}$ is a mfld of this dim.

__Def__: Floer product := linear map given by

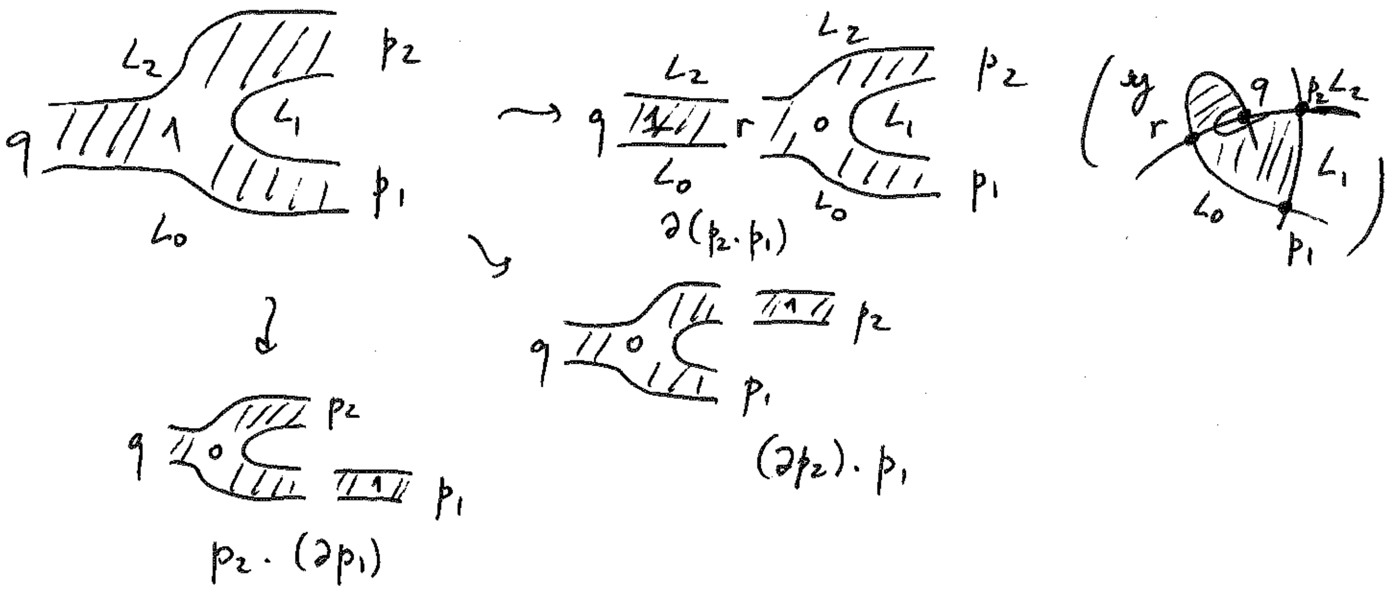
$$p_2 \cdot p_1 = \sum_{\substack{q \in \mathcal{X}(L_0, L_2) \\ [u] \,\mid\, \mathrm{ind}([u]) = 0}} \# \mathcal{M}(p_1, p_2, q, [u], J)\, T^{\omega([u])} q$$

__Prop__: If no disk bubbling (eg $[\omega] \cdot \pi_2(M, L_i) = 0$ $\forall i$), then

$\partial(p_2 \cdot p_1) = \pm (\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1)$ (Leibniz rule)

and the induced product $HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2)$ is independent of chosen J (& Ham perturbs) __and__ associative (unlike the product on CF^*).

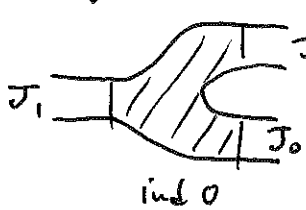
<u>Pf of Leibniz rule:</u> considering index 1 $\mathcal{M}(p_1, p_2, q, [u], J)$, have a 1-mfld which compactifies to a mfld w/ bdry (we've assumed no sphere / disk bubbling).



Signed count of bdry pts of compactified index 1 $\mathcal{M}$ is zero

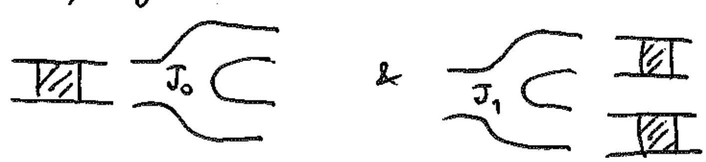
$\Rightarrow \quad \pm\, \partial(p_2 \cdot p_1) \pm (\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1) = 0$. ∎

Changing J (& Ham perturb), can study a continuation operation



Let 

move left or right. At ends, get



there might <u>also</u> be solutions 

These solutions can be used to write a homotopy between the product for J_0 & the product for J_1.

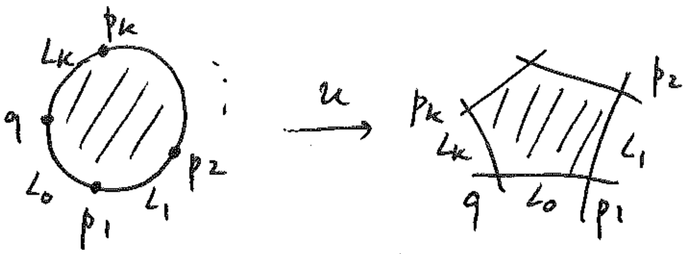
Associativity up to homotopy

Higher products: $\exists$ sequence of operations

$$\mu^k : CF^*(L_{k-1}, L_k) \otimes \cdots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k)[2-k].$$

$\mu^1 = \partial$, $\mu^2 =$ product.

μ^k counts J-holo disks

$\Big\{$ output q  , $\bar\partial_J u = 0 \Big\} / Aut(D^2)$

There's a $(k-2)$-dim'l family of such domains.

Expected dim $\mathcal{M}(p_1, \dots, p_k, q, [u], J) = \mathrm{ind}([u]) + k - 2$

$$\mu^k(p_k, -, p_1) = \sum_{\substack{q \in \mathcal{X}(L_0, L_k) \\ [u] \,|\, \mathrm{ind}[u] = 2-k}} \# \mathcal{M}(p_1, -, p_k, q, [u], J) \cdot T^{\omega([u])} q$$

Prop: If $[\omega] \cdot \pi_2(M, L_i) = 0 \ \forall i$, then A_∞-relations hold:

$$\sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} (-1)^* \mu^{k-\ell+1}(p_k, -, \mu^\ell(p_{j+\ell}, -, p_{j+1}), p_j, -, p_1) = 0$$

$$* = j + \deg(p_j) + - + \deg(p_1).$$

$k=1$: $\mu^1(\mu^1(p_1)) = 0$ ✓

$k=2$: Leibniz rule ✓

$k=3$: $\pm \mu^2(\mu^2(p_3, p_2), p_1) \pm \mu^2(p_3, \mu^2(p_2, p_1)) =$

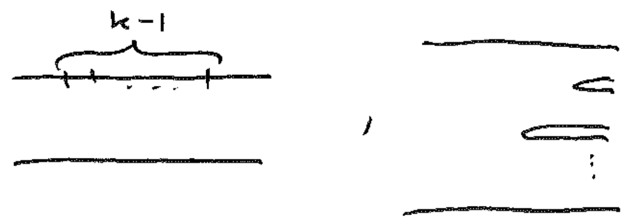
$= \pm \mu^1(\mu^3(p_3, p_2, p_1)) \pm \mu^3(\mu^1(p_3), p_2, p_1) \pm \mu^3(p_3, \mu^1(p_2), p_1)$

"μ^2 associative, up to homotopy given by μ^3". $\quad + \mu^3(p_3, p_2, \mu^1(p_1))$

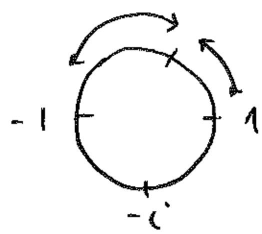
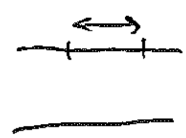
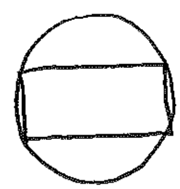
Pf of A_∞ - relation: In index $3-k$,

$\mathcal{M}(p_1, \dots, p_k, q, [u], J)$ is a 1-mfld, w/ compactification.

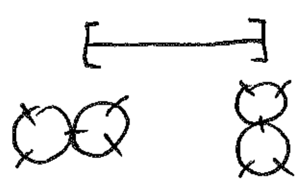
$\mathcal{M}_{0,k+1} = \Big\{$  $k+1$ pts on $\partial D^2 \Big\} / \mathrm{Aut}(D^2)$

Other models: 

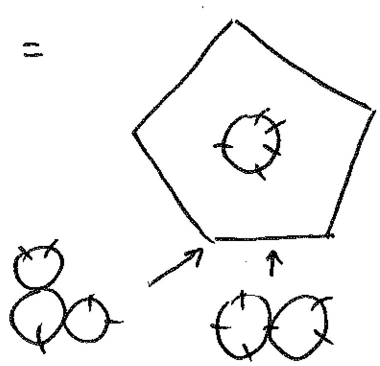
This space is contractible, w/ natural compactification to a polytope: the Stasheff associahedron.

Eg: $k=3$:  or  or  (disk w/ inscribed horizontal rectangle)

or 

$\overline{\mathcal{M}}_{0,4}$: 

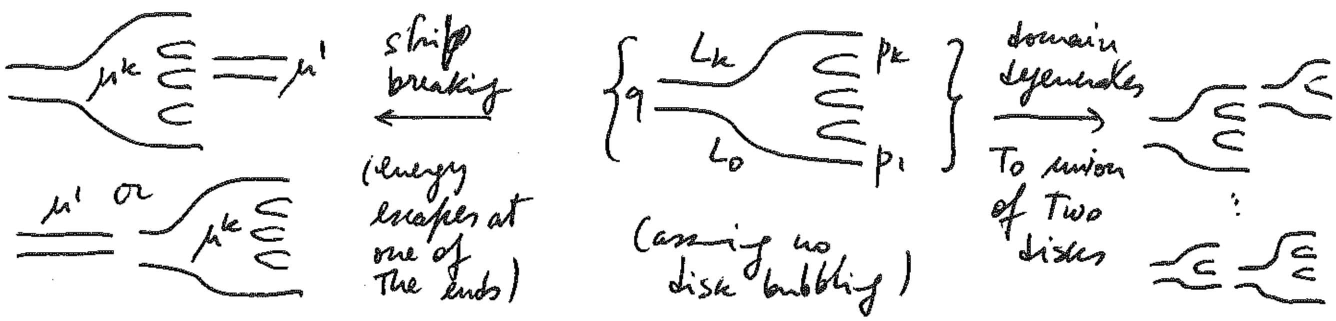
$k=4$: $\overline{\mathcal{M}}_{0,5} =$



If $\mathcal{M}(p_1, \dots, p_4, q, [u], J)$ is 1-dim,
then generically have 1-param family of domains:

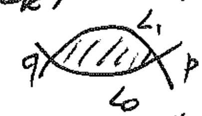
Codim 1 faces = pairs of disks
(generically, avoid higher codim faces)

So, the boundary of a 1-dim $\mathcal{M}(p_1, -, p_k, q, [u], J)$ is



(Trace energy escaping and breaking of domain).
terms w/ μ^1 terms w/o μ^1

Note: Naively, $\exists$ cyclic symmetry $CF^*(L_0, L_k) \simeq CF^{n-*}(L_0, L_k)^\vee$.



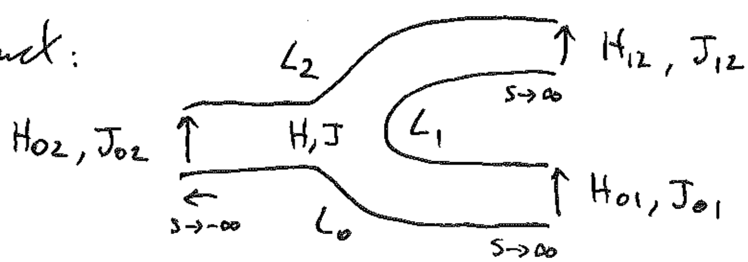
For $\langle \mu^k(p_k, \dots, p_1), q \rangle$, $\exists$ cyclic symmetry, up to homotopy.

$\rightsquigarrow$ Fukaya categories are Calabi-Yau A_∞-categories.

Hamiltonian perturbations

We defined $(CF^*(L_0, L_1), \partial)$ using H_{L_0,L_1} & J_{L_0,L_1}.

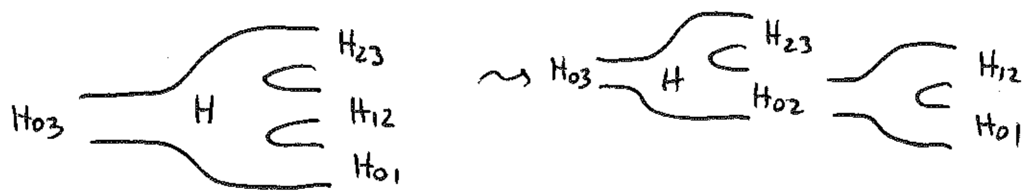
For product:



Choose strip-like ends : charts $z = s+it$ near punctures.

Eq: $\frac{\partial u}{\partial s} + J^t \left(\frac{\partial u}{\partial t} - X_{H^t} \right) = 0$ near ends.

This is the Floer eq w/ correct perturbations near ends.
That's it for strip-breaking.
What about <u>domain-breaking</u>?



H must vary differently w/ different domains!

<u>Thm</u> (Seidel): $\exists$ inductive procedure for constructing consistent families of (H, J).

The set of choices at each step is contractible (see Seidel's book).
This uses fact that associahedron is contractible.

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<u>Fukaya category</u>
(M, ω) cpct or w/ reasonable behavior at infinity.
Obj's = compact Lagrangian submflds, unobstructed (no holom discs), + spin structures (for char $\neq 2$),
(+ gradings if $2c_1(M) = 0$ and want $\mathbb{Z}$-grading),
+ local systems ("unitary" flat bundle / L).