To summarize: assumption $\int \omega = 0$ on disks w/ bdry on L_0 or L_1

$\Rightarrow$ no disk / sphere bubbling.

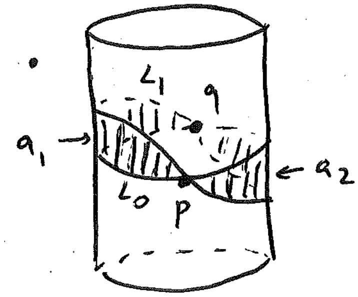
Then, if $\mathrm{ind}([u]) = 2$,

$$\partial \overline{\mathcal{M}(p,q; J, [u])} = \coprod_{\substack{r \in \mathcal{X}(L_0, L_1) \\ [u] = [u_1] \# [u_2]}} \mathcal{M}(p, r, J, [u_1]) \times \mathcal{M}(r, q, J, [u_2])$$

$$\underset{2}{\mathrm{ind}([u])} = \underset{1}{\mathrm{ind}([u_1])} + \underset{1}{\mathrm{ind}([u_2])}$$

$\underline{\partial^2 = 0}$: coeff of q in $\partial^2 p$ counts $p \to r \to q$, whose total # is 0.

$\underline{\text{Ex}}$:

- 

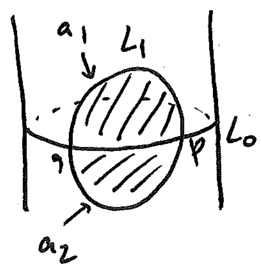
$a_i : p \to q$

$\partial(p) = (T^{a_1} - T^{a_2})\, q$

$\partial(q) = 0$

$\Rightarrow HF^*(L_0, L_1) \cong \begin{cases} \text{2-dim}, & \text{if } a_1 = a_2 \\ 0, & \text{if } a_1 \neq a_2 \end{cases}$

(can displace L_0 from L_1 iff $a_1 \neq a_2$)

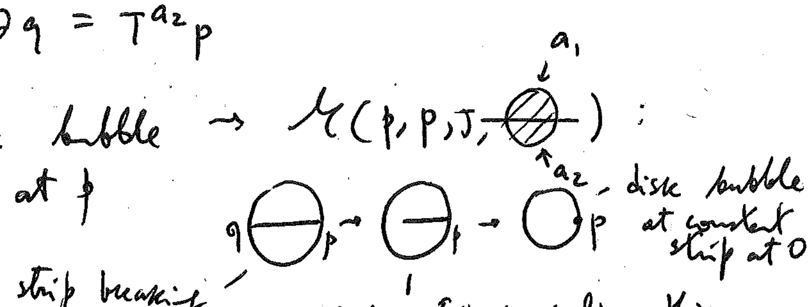
- 

$\partial p = T^{a_1} q$

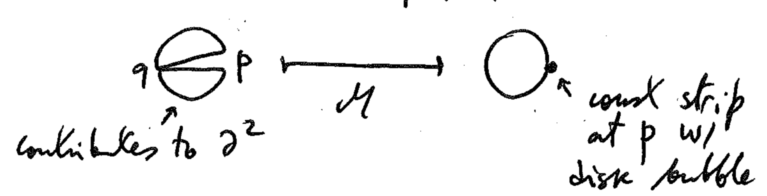
$\partial q = T^{a_2} p$

$\Rightarrow \partial^2 \neq 0$

disk bubble at p $\to$ $\mathcal{M}(p, p, J, \;)$:



$\exists!$ holo strip $p \to p$ like this



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Lagrangian Floer homology

$L_0, L_1 \subset (M,\omega)$ Lagrangian, $\quad H_{t\in[0,1]} \in C^\infty([0,1]\times M, \mathbb{R})$

$\qquad\qquad\qquad\qquad\qquad\qquad J_{t\in[0,1]}$ ω-compatible almost cx str

$$\rightsquigarrow \quad CF^*(L_0,L_1) = \bigoplus_{p\in\mathcal{X}(L_0,L_1)} \Lambda\cdot p$$

$\mathcal{X}(L_0,L_1) = \{$ time-1 trajectories of X_H starting at L_0 and ending at L_1 $\}$

$\qquad\cong \phi_H^1(L_0)\cap L_1$.

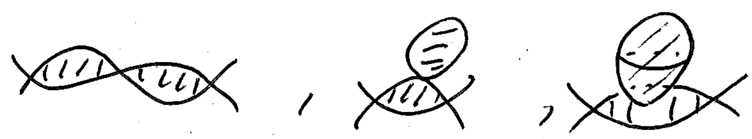
Given $p,q\in\mathcal{X}(L_0,L_1)$,

$$\mathcal{M}(p,q,J,[u]) = \left\{ u : \underset{s}{\mathbb{R}}\times\underset{t}{[0,1]} \to M \;\middle|\; q \uparrow \overset{L_1}{\underset{L_0}{/\!/\!/\!/}} \uparrow p,\ \frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - X_{H_t}\right) = 0 \right\} \Big/ \mathbb{R}$$

When the linearized operator is surjective,

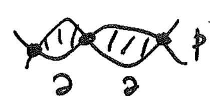
$\mathcal{M}$ is a manifold of $\dim_{\mathbb{R}} = \mathrm{ind}([u]) - 1$,

oriented given spin structures.

Compactification adds 

$$\partial(p) = \sum_{\substack{q \\ [u]:\,\mathrm{ind}=1}} \#\mathcal{M}(p,q,J,[u])\ T^{E(u)}\, q$$

<u>Thm</u> (Floer): If no bubbling (eg $[\omega]\cdot\pi_2(M,L_i)=0$), then $\partial^2=0$ and $HF^*(L_0,L_1) = H^*(CF,\partial)$ is indept of chosen (H,J), invt wrt Hamiltonian isotopy.

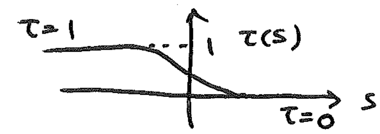
<u>Recall</u>: $\partial^2=0$: $\partial\left(\overline{\mathcal{M} \text{ of index 2 strips}}\right) =$ 

<u>Invariance</u> : via <u>continuation maps</u>. Given a family $(H_\tau, J_\tau)_{\tau\in[0,1]}$, build a chain map

$$\phi : CF^*(L_0, L_1 ; H_0, J_0) \to CF^*(L_0, L_1 ; H_1, J_1)$$

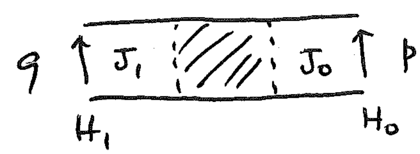
by counting solutions to

$$\frac{\partial u}{\partial s} + J_{\tau(s),t}\left(\frac{\partial u}{\partial t} - X_{H_{\tau(s),t}}\right) = 0$$



Get $\tilde{\mathcal{M}} = \{\text{solutions}\}$, $\dim \tilde{\mathcal{M}} = \operatorname{ind}[u]$ (no $\mathbb{R}$-action now)

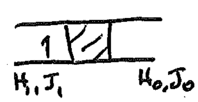
$$\phi(p) = \sum_{\substack{q \\ \operatorname{ind}[u]=0}} (\# \tilde{\mathcal{M}})\, T^{E(u)}\, q$$



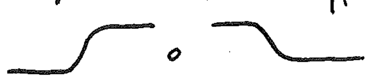
This is a <u>chain map</u>:

$\phi \circ \partial_0$ $\qquad\qquad$ $\partial_1 \circ \phi$

\ boundaries of /

$\tilde{\mathcal{M}}_{\text{index } 1}$ 

Homotopy of homotopies $\Rightarrow$ different ϕ's are homotopic.

In particular,  $\sim$ id

$\Rightarrow \phi$ is a <u>quasi-isomorphism</u>.

(constructing all these maps relies on space of auxiliary data being contractible)

<u>Index / grading</u>

<u>Maslov index</u> :

$$LGr(n) = \{\text{Lagr planes} \subset (\mathbb{R}^{2n}, \omega_0)\} = U(n)/O(n) \xrightarrow{\det^2} S^1$$

Lagrangian Grassmanian

convention (because take unoriented Lagr planes)

$\pi_1(\mathcal{L}Gr(n)) \simeq \mathbb{Z}$, $H^1(\mathcal{L}Gr(n)) \simeq \mathbb{Z}$

$\rightsquigarrow$ Maslov index of a loop of Lagr planes $\in \mathbb{Z}$.

This measures the rotation of the loop of Lagrs.

Ex.: $u : (D^2, \partial D^2) \to (M, L)$

$u^* TM$ can be trivialized ~~as~~ as a (symplectic / complex) vector bdle
$\overset{\|}{\mathbb{R}^{2n}}$

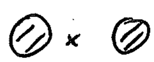
$(u|_{\partial D^2})^* TL$ gives a loop in $\mathcal{L}Gr(n)$.

$\mu =$ Maslov index $(u) \in \mathbb{Z}$

Ex: In $\mathbb{R}^2$:

because take unoriented planes, these two are the same (took $\det^2$...)

$\Rightarrow \mu = 2$ in this case

Note: μ is additive under
- connect sums 
- products 

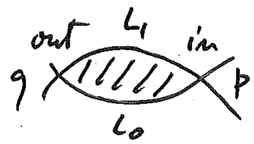
These properties + example above + ~~...~~ μ(constant disk) $= 0$ are enough to compute μ in many examples.

- $\mu\underset{\text{disk}}{([u])} = 2\, c_1(TM, TL) \cdot [u]$ (factor 2 from $\det^2$)

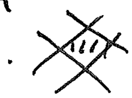
  totally real

  $c_1(TM,TL) \in H^2(M,L;\mathbb{Z})$, $[u] \in H_2(M,L;\mathbb{Z})$

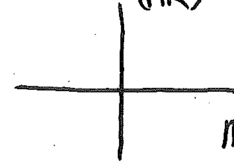
- how about $\mu(\text{strip})$?



More generally, for polygons.



Can take local coords near p st



there is a <u>prefered way</u> to homotope one to the other, namely rotate clockwise: get "canonical short path" from T_pL_0 to T_pL_1 : $(e^{-i\theta}\mathbb{R})^n$, θ from 0 to $\frac{\pi}{2}$

This is <u>independent of choices</u> of <u>local trivialization</u>.

For the whole strip:

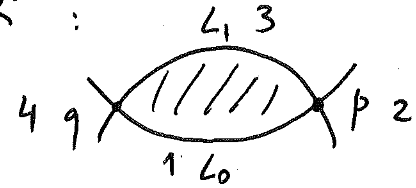
$$T_qL_0 \xrightarrow[\text{along } \partial u]{TL_0} T_pL_0 \xrightarrow[\text{short path}]{\text{canonical}} T_pL_1 \xrightarrow[\text{along } \partial u]{TL_1} T_qL_1 \xrightarrow[\text{short path}]{\text{inverse of canonical}} T_qL_0$$

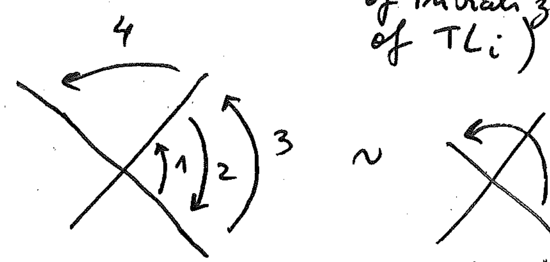
Trivializing u^*TM, get a loop in $\mathcal{L}Gr(n)$.

$\text{ind}([u]) =$ Maslov index of this loop.

(result is indep also of choices of trivializations of TL_i)

<u>Ex</u>: In $\mathbb{R}^2$:

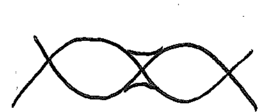




$\mu = 1$!

This matches fact that strip is unique, up to $\mathbb{R}$-translation (Riem mapping Thm).

<u>Note</u>: this is additive under



Q: When is there a $\mathbb{Z}$-grading on CF^*?

This is so that $p \in \mathcal{X}(L_0, L_1) \rightsquigarrow \deg(p) \in \mathbb{Z}$

$\operatorname{ind}([u]) = \deg(q) - \deg(p)$

$\partial : CF^k \to CF^{k+1}$

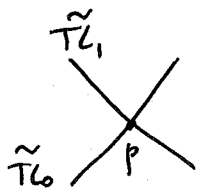
For index to be indep't of $[u]$, need $\underset{\swarrow \det^2}{2} c_1(TM) = 0$

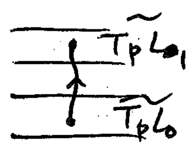
$2c_1(TM) = 0 \iff \exists\ \mathbb{Z}$-cover $\begin{array}{c} \widetilde{LGr}(TM) \\ \downarrow \\ LGr(TM) \\ \downarrow \\ M \end{array}$ (univ cover at each fiber)

$H^1(L_i; \mathbb{Z})$
Ψ
Maslov class $\mu(L_i) = 0 \iff L_i$ ~~lifts~~, which always lifts to $LGr(TM)$, also lifts to $\widetilde{LGr}(TM)$.

In that case, $\exists$ graded lift of TL.
This is unique up to additive shift $\rightsquigarrow$ shift functors on $\mathcal{F}(M)$!

Note: these notions depend on the choice of trivialization of $\Lambda^{top}(TM)$.

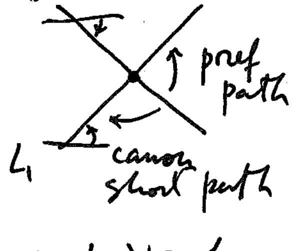




$\widetilde{LGr}(T_pM)$

To get $\deg(p)$, compare preferred path $T_pL_0 \to T_pL_1$ with canonical short path

Ex:



$\rightsquigarrow \deg(p) = 1$

Note: Even if don't have $\mathbb{Z}$-gradings as above, always have $\mathbb{Z}/2$-grading on CF^* (bc of $\det^2$...).

$HF^*(L,L)$?

Local case: $L =$ zero section $\subset T^*L$.

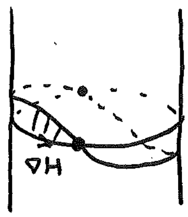
Pick $\varepsilon H : L \to \mathbb{R}$ Morse function (lift to $T^*L \to \mathbb{R}$)

$$\phi^1_{\varepsilon H}(L) = \text{graph}(\varepsilon\, dH)$$

$$\mathcal{X}(L, L, \varepsilon H) = \text{Crit}(H)$$

Clever choice of J: along zero-section, want

$J(\nabla H) = \pm X_H$ (want identify TL w/ T^*L fibers, which is same as choosing a metric)



Morse trajs on L $\quad \overset{u(s,t) = (\gamma(s), 0)}{\underset{\varepsilon \to 0}{\longleftrightarrow}} \quad \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{\varepsilon H}\right) = 0$

$\dot{\gamma}(s) = -\varepsilon \nabla H(\gamma(s))$

<u>Floer's Thm:</u> $HF^*(L,L) \cong HMorse^*(H) = H^*(L)$ in T^*L.

Case of general (M,ω): understand Floer sols as $\varepsilon \to 0$

(Floer, Fukaya-Oh, Biran-Cornea, ...).

As $\varepsilon \to 0$, ~~sol~~ strips look like

thin part, closer and closer to a grad flow line of H

Solutions converge to union of

$\begin{cases} \text{gradient flow lines of } H \\ J\text{-holo disks w/ boundary on } L \end{cases}$

If no disks (eg $\omega \cdot \pi_2(M,L) = 0$), then $HF^*(L,L) \simeq H^*(L)$.

If there <u>are</u> disks, might not even have $\partial^2 = 0$!

If L <u>monotone</u> ($\mu(\text{disks}) = K\, \omega(\text{disks})$ for some $K > 0$

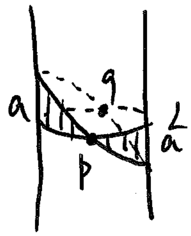
OR $\begin{cases} \text{no disks of } \mu < 2 \text{ (except constants)} \\ \text{AND } \mu = 2 \text{ disks are regular} \end{cases}$, then $\boxed{\partial^2 = 0}$ on $CF^*(L,L)$.

In such cases, can *filter* CF^* by Maslov index.

Get a *spectral sequence (Oh)* $H^*(L) \Rightarrow HF^*(L,L)$

(not $\mathbb{Z}$-graded, if non-trivial disks)

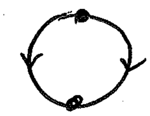
Ex: •



in T^*S^1

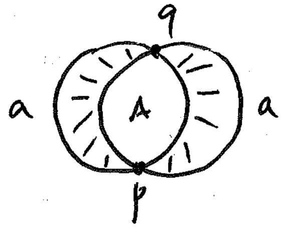
$$\partial p = (T^a - T^a)\, q = 0$$
$$\partial q = 0$$

in Morse theory:



$$HF^*(L,L) = H^*(L)$$

•

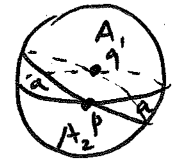


in $\mathbb{C}$

$$\partial p = (T^a - T^a)\, q = 0$$
$$\partial q = T^A\, p$$

$HF^* = 0$

•



equator

in $\mathbb{CP}^1$

$$\partial p = (T^a - T^a)\, q = 0$$
$$\partial q = (T^{A_1} - T^{A_2})\, p = 0$$

$HF^*(L,L) = H^*(S^1)$ if $A_1 = A_2$