Q: How to build $X^\vee$ geometrically?

$X^\vee$ = moduli space of points of $X^\vee$

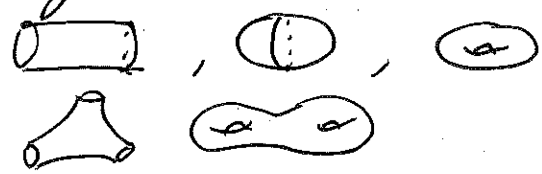
$\mathcal{O}_p$ skyscraper $\overset{HMS}{\longleftrightarrow}$ L_p,

$HF^*(L_p, L_p) = \mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong \Lambda(\mathbb{C}^n) \cong H^*(T^n)$ (so, L_p should be T^n)

$\rightsquigarrow$ $X^\vee$ = moduli space of T^n $(+\dots)$ in $\mathcal{F}(X)$.

This is related to SYZ conj (96), and gives a justification for why $X^\vee$ is a combinative space.

Plan:
1) Lagr HF & Fuk categs
2) HMS in 1d :



3) SYZ & constr of mirrors
4) In progress.

References: Abeginner's guide to Fuk categories (Auroux)
Paul Seidel's book (part 2)
Sheridan's Paris notes.

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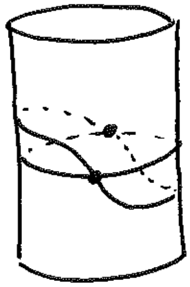
Lagrangian Floer cohomology

(M, ω) sympl (compact or cvx at ∞)
$\cup$
L cpct closed Lagrangian

Thm (Floer): Assume $\int \omega = 0$ over any disk w/ $\partial D^2 \subset L$.
Let $\psi \in \mathrm{Ham}(M, \omega)$ (flow of X_H, $\omega(X_H, \cdot) = dH$).
Assume $\psi(L) \pitchfork L$. Then, $|\psi(L) \cap L| \geq \sum_i \dim H^i(L; \mathbb{Z}/2)$

Ex:



$\psi \in \mathrm{Ham}$: area b/w L and $\psi(L)$ is zero.

$|\psi(L)\cap L| \geq 2$

(not true if $\psi \in \mathrm{Symp}$)

(not true if )

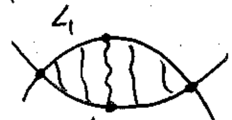
Will have :

- $CF^*(L_0, L_1)$ freely gen'd by $L_0 \cap L_1$
- ∂ st $\partial^2 = 0$ $\rightsquigarrow$ $HF^*(L_0,L_1) = \ker\partial/\mathrm{im}\,\partial$
- If L_1, L_1' Ham isotopic, then $HF^*(L_0, L_1) \simeq HF(L_0, L_1')$
- $HF^*(L,L) \simeq H^*(L)$.

Then, $|\psi(L)\cap L| = \operatorname{rank} CF^*(L,\psi(L)) \geq \operatorname{rank} HF^*(L,\psi(L)) = \operatorname{rank} HF^*(L,L) =$
$= \operatorname{rank} H^*(L)$ ∎

(This proves Floer's thm).

Morally, HF^* is the Morse cohomology of "action functional" on (cover of) space of paths $[0,1] \to (M, L_0, L_1)$, whose crit pts are constant paths



This is hard to do rigorously.

Floer: replace grad flow on path space w/ a PDE on $\mathbb{R}\times[0,1]$.

Coeff field:

$\Lambda_{\mathbb{K}}$ = Novikov field over base field $\mathbb{K}$ ($\mathbb{Z}/2$, $\mathbb{Q}$, $\mathbb{C}$, ...).

$\Lambda_{\mathbb{K}} = \{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = \infty \}$

$\llcorner$ keeps track of areas of holo curves

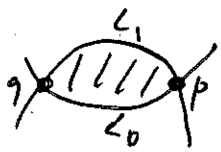
Assume $L_0 \pitchfork L_1$. $\mathcal{X}(L_0,L_1) := L_0 \cap L_1$.

$$CF^*(L_0,L_1) = \bigoplus_{p \in \mathcal{X}(L_0,L_1)} \Lambda_{\mathbb{K}}\, p.$$

(M,ω) carries a compatible almost-complex structure J ($J^2=-1$, $\omega(\cdot, J\cdot)$ Riem. metric).

Study maps $u : \underset{s}{\mathbb{R}} \times \underset{t}{S^1} \to M$ st

- $\dfrac{\partial u}{\partial s} + J \dfrac{\partial u}{\partial t} = 0$
- $u(s,0) \in L_0$
- $u(s,1) \in L_1$
- $\lim_{s\to\infty} u(s,t) = p \in \mathcal{X}(L_0,L_1)$
- $s \to -\infty$ $\quad q$



Fix htpy class $[u] \in \pi_2(M, L_0 \cup L_1)$.

$$E(u) = \iint_{\mathbb{R}\times[0,1]} \frac{1}{2}\left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \right) ds\,dt = \int_{\mathbb{R}\times[0,1]} u^*\omega = \int_{[u]} \omega$$

This quantity is <u>homological</u>, so invariant under perturbations of u in same htpy class.

The linearization of the PDE is <u>Fredholm</u>. Call it $D_{\bar\partial}$.

index $(\bar\partial)$ = ind $([u])$ = expected dim of space of sols.

(unlike in Morse case, this index depends on $[u]$; it's given by a spectral flow).

If the linearized operator $D_{\bar\partial}$ is surjective at every solution (true for generic J), then the space of sols

$\hat{\mathcal{M}}(p,q,J,[u])$ is a smooth mfld of dim = ind $([u])$.

$\mathcal{M} := \hat{\mathcal{M}}/\mathbb{R}$ (translation in s direction)

If $\operatorname{ind}([u]) = 1$, then $\mathcal{M}$ is a discrete set, actually <u>finite</u> (Gromov compactness).

<u>Def</u>: $\boxed{\partial(p) = \sum_{q \in \mathcal{X}(L_0, L_1)} \# \mathcal{M}(p, q; J, [u]) \, T^{\omega([u])} q}$,

$[u]$ st $\operatorname{ind}([u]) = 1$ signed count if can orient $\mathcal{M}$

<u>Rmk</u>: 1) In general, given p, q, could have ∞ly many $[u]$ st ind $= 1$.
<u>But</u> Gromov cptness $\Rightarrow$ only finitely many have $\omega([u]) < K$ $\forall K$

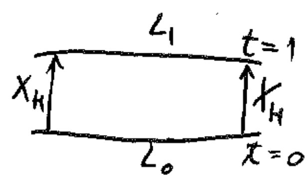
2) There are settings where have a priori finiteness & energy estimates.
Eg: if $(M, \omega = d\theta)$ exact, and L_0, L_1 exact ($\theta|_{L_i}$ exact), then $\omega([u])$ is determined by p & q. Can get rid of T.

3) $\mathcal{M}$ can be oriented if L_0 & L_1 are oriented and spin.
(spin stru related to orientations of path spaces)
(see lecture of Sheridan's on this)

4) Transversality / compactness:
↳ ensure $\mathcal{M}$ is smooth, by ensuring $D_{\bar\partial}$ is onto
Pick J generic, possibly t-dependent. If $L_0 \pitchfork L_1$, then ✓.
Pick $H : M \times [0,1] \to \mathbb{R}$ Hamilt. perturb. (generic). Study

$$\frac{\partial u}{\partial s} + J(t, u(s,t)) \left(\frac{\partial u}{\partial t} - X_H(t, u(s,t)) \right) = 0 .$$

Generators: $\mathcal{X}(L_0, L_1) = \{ \gamma : [0,1] \to M \mid \dot\gamma(t) = X_H(\gamma(t)), \gamma(0) \in L_0, \gamma(1) \in L_1 \}$

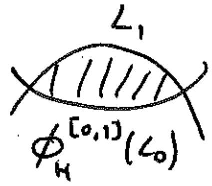


(useful to wrap)
see Seidel's book for this in Fuk.

Rescaling trick: Take $v(s,t) = \phi_H^{[t,1]}(u(s,t))$, which solves

$$\frac{\partial v}{\partial s} + (\phi_* J)\left(\frac{\partial v}{\partial t}\right) = 0 \qquad \left(\phi_H^{[0,1]}(\gamma(0)) = \gamma(1)\right)$$

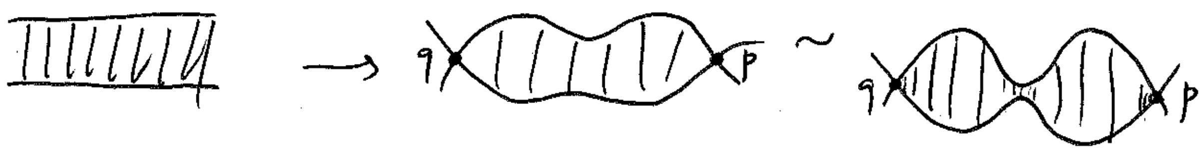
no Hamiltonian term!



Compactness, $\partial^2 = 0$

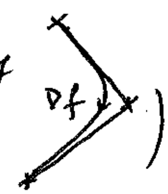
Gromov compactness: a sequence $u_i \in \mathcal{M}(p,q,J,[u])$ has a subsequence converging to a finite union of

- J-holo (perturbed) strips $\in \mathcal{M}(p_i, q_i, \dots)$
- J-holo disks w/ bdry on L_0 or L_1
- J-holo spheres



strip-breaking

(similar to Morse breaking



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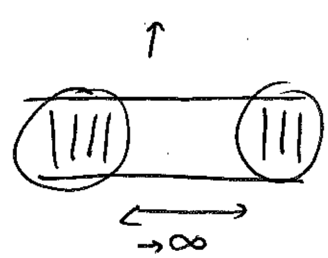


Worse: disk-bubbling:

"codim 1"

(doesn't exist if assume no disks w/ $\int \omega = 0$, w/ bdry on L_0, L_1)

sphere bubbling:

"codim 2"

(also ruled out, since can relate a sphere to a disk w/ bdry on L_i)

To summarize: assumption $\int \omega = 0$ on disks w/ bdry on L_0 or L_1

$\Rightarrow$ no disk / sphere bubbling.

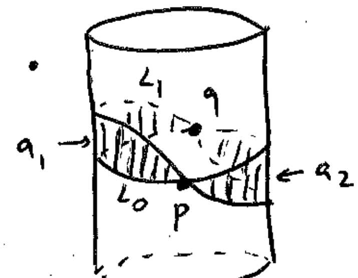
Then, if $\operatorname{ind}([u]) = 2$,

$$\partial \overline{\mathcal{M}(p,q;J,[u])} = \coprod_{\substack{r \in \mathcal{X}(L_0,L_1) \\ [u]=[u_1]\#[u_2]}} \mathcal{M}(p,r,J,[u_1]) \times \mathcal{M}(r,q,J,[u_2])$$

$$\underset{2}{\operatorname{ind}([u])} = \underset{1}{\operatorname{ind}([u_1])} + \underset{1}{\operatorname{ind}([u_2])}$$

<u>$\partial^2=0$</u>: coeff of q in $\partial^2 p$ counts $p \to r \to q$, whose total # is 0.

<u>Ex</u>:

- 

$a_i : p \to q$

$\partial(p) = (T^{a_1} - T^{a_2})\, q$

$\partial(q) = 0$

$\Rightarrow HF^*(L_0,L_1) \cong \begin{cases} 2\text{-dim}, & \text{if } a_1 = a_2 \\ 0, & \text{if } a_1 \neq a_2 \end{cases}$

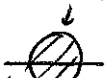
(can displace L_0 from L_1 iff $a_1 \neq a_2$)

- 

$\partial p = T^{a_1} q$

$\partial q = T^{a_2} p$

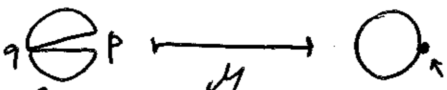
$\Rightarrow \partial^2 \neq 0$

disk bubble at p $\to$ $\mathcal{M}(p,p,J,$  $)$:

strip breaking

disk bubble at constant strip at 0

$\exists!$ hol. strip $p \to p$ like this



contributes to ∂^2

const strip at p w/ disk bubble (doesn't contribute to ∂^2)