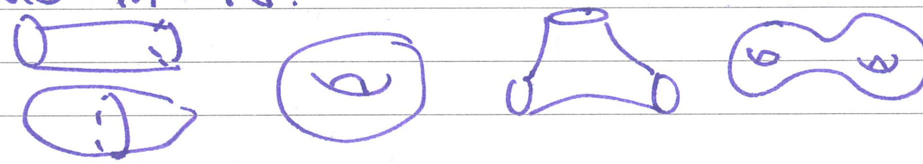
Eilenberg I

Strominger-Yau-Zaslow

this is consistent w/ SYZ conj of '96

Plan:
1) LF & Fukaya categories
2) HMS in 1d.



3) SYZ philosophy & construction of mirrors
4) stuff in progress / hopes for future.

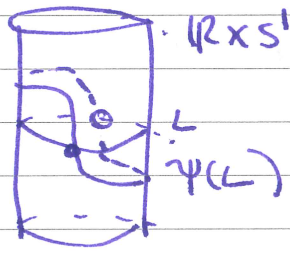
9/15/16 Lagrangian Floer cohomology

(M, ω) symplectic
$\cup$
L Lagrangian

thm (Floer): Assume $\int_D \omega = 0$ for any disk D w/ $\partial D^2 \subset L$. Then let $\psi \in \mathrm{Ham}(M, \omega)$ & $\psi(L) \cap L$ transverse. Then

$$|\psi(L) \cap L| \geq \sum_i \dim H^i(L, \mathbb{Z}/2).$$

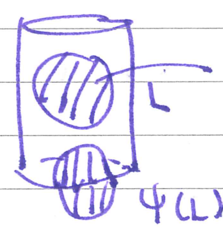
ex:



$|\psi(L) \cap L| = 2$

Note: $\psi \in \mathrm{Ham} \Rightarrow$ signed area between L & $\psi(L)$ is 0

not true if $\psi \in \mathrm{Symp}$



bounds a disk of pos. area

① $CF^*(L_0, L_1)$ is freely generated by $L_0 \cap L_1$, ∂, $\partial^2 = 0$ $HF^* = \ker \partial / \mathrm{im}\, \partial$

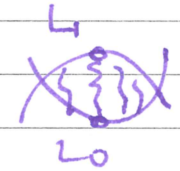
③ $HF^*(L, L) \simeq H^*(L)$.

② If L_1' is Ham iso to L_1 then $HF^*(L_0, L_1) \simeq HF^*(L_0, L_1')$

then $|\Psi(L) \cap L| = rk\, CF^*(L, \Psi(L)) \geq rk\, HF^* = rk\, H^*(L)$

how do we define this?

Intuition: Morally, HF^* is the Morse theory of the action functional on (a cover of) the space of paths $[0,1] \to (M; L_0, L_1)$



whose crit pts are const. paths

gradient flow.

need to translate proof of $\partial^2 = 0$ in Morse theory to the language of pseudohol. curves.

coeff field: Novikov field over $\mathbb{K}$ ($\mathbb{Z}/2, \mathbb{C}, \mathbb{Q}$)

$\mathbb{K}$ little

big $\Lambda_{\mathbb{K}} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = +\infty \right\}$

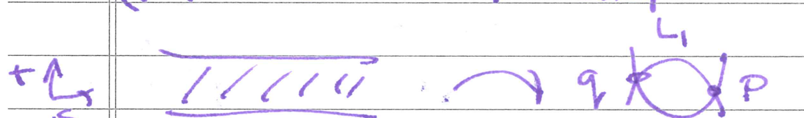
like a formal power series $\rightsquigarrow$ just w/ real exp.

This will let us encode symplectic area of trajectories

$$CF^*(L_0, L_1) = \bigoplus_{p \in \mathcal{X}(L_0, L_1)} \Lambda_{\mathbb{K}}\, p$$

$\uparrow$ intersections

(M, ω) carries a compatible acs J $\left[J^2 = -id,\ \omega(\cdot, J\cdot) \text{ Riem metric} \right]$



$\mathbb{R} \times [0,1] \xrightarrow{u} M$ s.t. $u(s,0) \in L_0$, $u(s,1) \in L_1$

s t

$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0$

$\left. \begin{array}{l} \lim_{s \to +\infty} u(s,t) = p \\ \lim_{s \to -\infty} u(s,t) = q \end{array} \right] \in \mathcal{X}(L_1, L_0)$

satisfying $du \circ j = J \circ du$.

Fix the htpy class $[u] \in \pi_2(M, L_0, L_1)$.

$$E(u) = \iint_{\mathbb{R}\times[0,1]} \frac{1}{2}\left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right) ds\, dt$$

$$= \int_{\mathbb{R}\times[0,1]} u^*\omega = \int_{[u]} \omega$$

→ indep of u wrt its htpy class by Stokes' b/c ω is ~~closed~~ closed.

The ~~$\bar\partial$~~ linearization of PDE $D_{\bar\partial}$ is Fredholm
we can calculate its index
$\text{ind}(\bar\partial) = \text{ind}([u])$.
is the expected dim of the space of sol.

If $D_{\bar\partial}$ is surjective at every sol (generic J) e.g. for
then the space of sol's
$\hat{\mathcal{M}}(p, q, J, [u])$ is
a smooth mfld of $\dim = \text{ind}([u])$.

$\mathcal{M} := \hat{\mathcal{M}}/\mathbb{R}$ (trans. in s dir)

If $\text{ind}([u]) = 1$, $\mathcal{M}$ is a discrete set, finite (Gromov compactness)

def: $$\partial(p) = \sum_{\substack{q \in \mathcal{X}(L_0, L_1) \\ [u];\ \text{ind}(u) = 1}} \#\mathcal{M}(p, q, J, [u])\, T^{\omega([u])}$$

signed count if can orient $\mathcal{M}$.

Remarks 1) Given T we could have infinitely many $[u]$ s.t. ind $=1$. But Gromov compactness says we must have only finitely many solutions if we also impose $\omega([u]) < K$ $\forall K$.

There are settings where we have a priori finiteness (b/c of a priori energy estimates).
e.g. if M is exact symplectic & L_0, L_1 are exact Lag's then $\omega([u])$ is determined by p & q. Thus we can get rid of T

2) $\mathcal{M}$ can be oriented if we assume L_0, L_1 are oriented & spin.

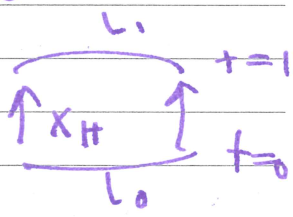
3) <u>transversality</u> & compactness.

↳ ensure $\mathcal{M}$ smooth by ensuring $D_{\bar\partial}$ onto
pick J generic, possibly t-dependent
If $L_0 \pitchfork L_1$ then ✓

Pick $H: M\times[0,1]\to\mathbb{R}$ Ham perturbation (generic)

$$\frac{\partial u}{\partial s} + J(t, u(s,t))\left(\frac{\partial u}{\partial t} - X_H(t, u(s,t))\right) = 0$$

$$\mathcal{X}(L_0, L_1) = \left\{ \gamma : [0,1] \to M \;\middle|\; \begin{array}{l} \dot\gamma = X_H(t) \\ \gamma(0) \in L_0 \\ \gamma(1) \in L_1 \end{array} \right\}$$



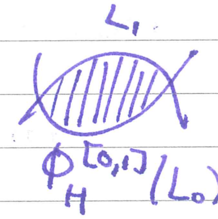
If noncompact might have a large perturbation at ∞ ⇝ will bring in geometry from boundary to the picture ... more on this later.

need to pick J & H together. See also Paul's book

$\phi_H^{[0,1]}(\gamma(0)) = \gamma(1) \in L_1$

$v(s,t) = \phi_H^{[t,1]}(u(s,t))$ so

$\frac{\partial v}{\partial s} + (\phi_* J)\left(\frac{\partial v}{\partial t}\right) = 0$

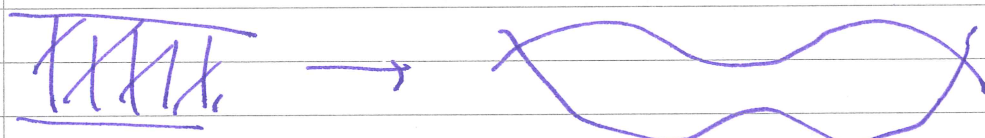


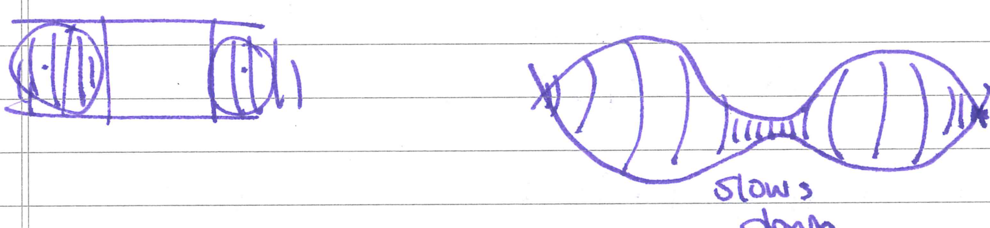
compactness, $\partial^2 = 0$

→ Gromov compactness ← In General set up

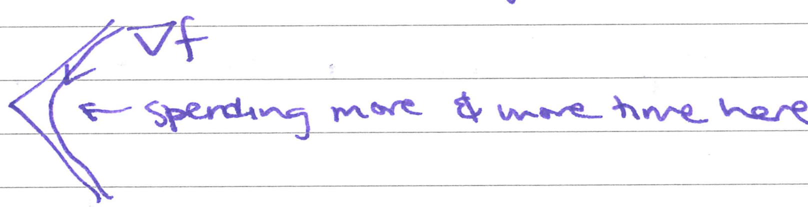
A seq. of $u_i \in \mathcal{M}(p, q, J, [u])$ has a subseq converging to a union of

- J hol (perturbed) strips $\in \mathcal{M}(p_i, q_i, \ldots)$
- J hol discs w/ bdry on L_0 or L_1
- J hol spheres



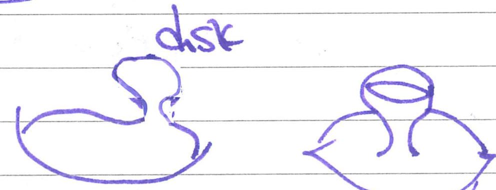


so a strip can converge to a chain of strips!



STRIP BREAKING.

bubbling:



derivative of map is going to ∞.

rescale at blow up of deriv: see a disk or sphere.

→ get strip breaking
sphere bubbling
disk bubbling] excluded if $\int_{\text{discs}} \omega = 0$. ☺

disc bubbling, horrid but a fact of life.
makes $\partial^2 \neq 0$

sphere bubbling, annoying but codim 2 so
shouldn't ~~effect~~ $\partial^2 = 0$.

strip breaking: good, like in Morse theory

$\mathcal{M}(p, q, [u], J)$ should be a 1D mfld.
$\uparrow$
index 2

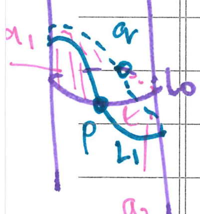
It admits a compactification by adding in bdry
which consists of broken strips.

If $\mathrm{ind}([u]) = 2$

$$\partial \overline{\mathcal{M}}(p, q, [u], J) = \coprod_{\substack{r \in \mathcal{X}(L_0, L_1) \\ [u] = [u_1] \# [u_2]}} \mathcal{M}(p, r, [u_1], J) \times \mathcal{M}(r, q, [u_2], J)$$

the index is additive provided we have transversality,
$\mathrm{ind}([u]) = \mathrm{ind}([u_1]) + \mathrm{ind}([u_2])$
$2 = 1 + 1$ ✓.

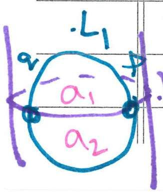
coeff of q in $\partial^2 p$ counts $p \to r \to q$.
total # $= 0$.



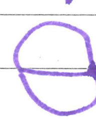
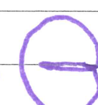
both $p \to q$
$\partial(p) = (T^{a_1} - T^{a_2})q$
$\partial(q) = 0$

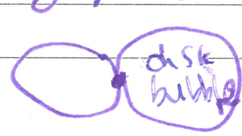
$HF^*(L_0, L_1) \cong \begin{cases} 2D & \text{if } a_1 = a_2 \\ 0 & \text{if } a_1 \neq a_2. \end{cases}$

so $\mathcal{M}(q, p, J) = \vdash\!\!-\!\!\dashv$.



$\partial p = T^{a_1} q$
$\partial q = T^{a_2} p$

$\partial^2 \neq 0$. get  &  slit

get 

as slit shrinks

good thinking

get  as slit is bigger