Denis Auroux 9/8

Eilenberg Lectures : Fukaya categories and mirror symmetry

Classif of Lagr submflds up to Ham. isotopy is a HARD proble.

$\mathbb{R}^4$ (G. Dimitroglou-Rizell '16? - only T^2_{cl} & T^2_{ch})

$\mathbb{CP}^2$: have some ideas about what could be a complete list.

$\mathbb{R}^6$: some is known, but still a lot unclear

Floer cohomology

When defined,

$HF^*(L,L) \cong H^*(L)$ if no disks of $\int\omega > 0$.

Eg: for $L \subset \mathbb{R}^{2n}$, $HF(L,L) = 0$

Cor : such L must bound disks of $\int\omega > 0$.

Fukaya A_∞-categ : higher compositions are homotopy data for failure of associativity.

Derived Fuk categ : triangulated structure
$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u] & \end{tikzcd}$$

Often, Fuk categs have generating objs G :

G (split) generates $\mathcal{F}(M)$ if every object is $\simeq$ (direct summand of) iterated mapping cone built from copies of G.

In that case, if understand $\mathrm{End}(G) =: \mathcal{A}$ as an A_∞-algebra, then understand arbitrary objects : $\mathcal{F}(M) \hookrightarrow \text{mod-}\mathcal{A}$

$T \mapsto \mathrm{Hom}(G,T)$

fully faithful embedding.

Rmk : classifying Lagrs up to Floer isom $\neq$ Ham isotopy.

Ex: • (T^2_{Cl}, ∇) some local syst is Floer-theoretically equivt to T^2_{Ch} in $\mathbb{P}^2$, but they are not Ham isot.

• Taking mapping cone of $L \to L'$ st $HF(L')=0$ gives another L'', which is Floer-equivt to L but may even have different topology.

Non-cpct case: Wrapped Fuk categ..
Fuk-Seidel categ...

Mirror symmetry: ~ 1990, topological string theory.
Calabi-Yau 3-folds come in pairs
$X^3_{\mathbb{C}}$, $K_X = \Omega^{3,0} \cong \mathcal{O}_X$ st $H^{p,q}(X) = H^{n-p,q}(X^\vee)$ $(n=3)$

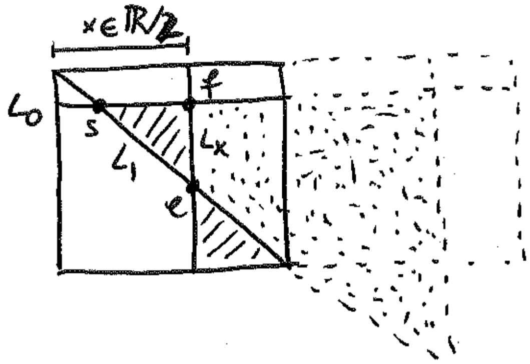
and Gromov-Witten invts of X can be predicted from Hodge period integral theory on $X^\vee$. (Givental, Lian-Liu-Yau).

Kontsevich: 1994 HMS: derived equivalence (slightly more...)
$D^\pi \mathcal{F}(X) \cong D^b Coh(X^\vee)$. (RHS is generated by v.bdles if $X^\vee$ smooth)

(the two versions should be related via HH^* - GPS).

Should thus have correspondeces
Lagr submflds of X $\longleftrightarrow$ coh sheaves on $X^\vee$
Lagr $\cap$ (connected by $\cancel{\text{hol discs}}$) $\longleftrightarrow$ Ext $(\cap, H^*_{\bar\partial})$.

Ex: T^2 (Polishchuk - Zaslow '98)



$$\begin{tikzcd} L_0 \arrow[r, "s"] \arrow[rr, bend right, "f"'] & L_1 \arrow[r, "e"] & L_x \end{tikzcd}$$

$$e \cdot s = \Big(\sum_{n\in\mathbb{Z}} T^{(n+x)^2/2}\Big) f$$

Mirror is elliptic curve $E = \mathbb{C}/\mathbb{Z} + i\tau\mathbb{Z}$

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"] \arrow[rr, bend right] & \mathcal{L}_1 \arrow[r, "e"] & \mathcal{O}_x \end{tikzcd} \text{ - pt} \qquad \mathcal{L}_1 \text{ deg 1 line bdle}$$

$$\Theta(\tau, \tau x) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n x \tau} = e^{-\pi i \tau x^2} \sum_{n \in \mathbb{Z}} \big(\underbrace{e^{2\pi i \tau}}_{T}\big)^{(n+x)^2/2}$$

HMS subsequently extended to | non-cpct
| non- CY

(involving wrapped, Fukaya-Seidel
matrix factorizations, categories of singularities)

<u>Another viewpoint</u>:

$\mathcal{F}(X) \hookrightarrow$ mod-$\mathcal{A}$, $\mathcal{A} = \mathrm{End}(G)$

If $X^\vee = \mathrm{Spec}\, R$, then $D^b\mathrm{Coh}(X^\vee) =$ mod-R.

In general, the algebraization of sympl geometry
$\Rightarrow$ non-commutative alg. geometry.

From this perspective, the remarkable fact is that often this gives an honest (commutative) algebraic space!
(In general, over Novikov field, but sometimes over $\mathbb{C}$, in which case MS is actually <u>involutive</u> - ie $(X^\vee)^\vee = X$.
People are actually thinking about sympl geometry over non-Archimedian spaces)

Q: How to build $X^\vee$ geometrically?

$X^\vee$ = moduli space of points of $X^\vee$

$\mathcal{O}_p$ skyscraper $\xleftrightarrow{\text{HMS}}$ L_p, $L_p = T^n$ Lagr.

$HF^*(L_p, L_p) = \mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong H^*(T^n)$

$\rightsquigarrow X^\vee$ = moduli space of T^n $(+\dots)$ in $\mathcal{F}(X)$.

This is related to SYZ conj (96), and gives a justification for why $X^\vee$ is a commutative space.

Plan:
1) Lagr HF & Fuk categs
2) HMS in 1d :

3) SYZ & constr of mirrors
4) In progress.

References: A beginner's guide to Fuk categories (Auroux)
Paul Seidel's book (part 2)
Sheridan's Paris notes.