Eilenberg I

9/8/16 Denis Auroux: Fukaya Categories & Mirror Symmetry

→ from perspective of LF

Lagrangian submflds in a symplectic mfld (M^{2n}, ω) are maximal isotropic submflds, L^n, $\omega|_L = 0$.

ex: $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i) \hookleftarrow \mathbb{R}^n_x, \mathbb{R}^n_y$] universal local model.

ex: $N \hookrightarrow T^*N$, $\omega_0 = \sum dp_i \wedge dq_i$ — zero section is Lag

$q_1, \ldots, q_n$ $\quad p_1, \ldots, p_n$

⇝ Weinstein's nhood thm says any Lag submfld in (M^{2n}, ω) has a nhood modelled on this.

Integrable systems, toric geometry

Lagrangians have interesting interection properties.
If $L \pitchfork L' = \{pt\}$
transverse,

early obs by Arnold: Lag's want to intersect more often than they have the right to!

Ham(M, ω) group of Hamiltonian diffeos.
$H \in C^\infty(M, \mathbb{R}) \rightsquigarrow \exists !$ VF X_H s.t. $\omega(X_H, \cdot) = dH$

Flow of Ham VF (time dep) gives $(\varphi^H)^* \omega = \omega$.

Ham isotopies "smallest" change we can make to the symplectic str... but nothing happens to the form. What if we move Lag's around this way?

if $\pitchfork$

$\downarrow$

Arnold Conj — often, $\forall \varphi \in \text{Ham}$, $\#(\varphi(L) \cap L) \geq \dim H^*(L)$

often: e.g. when L doesn't bound any discs w/ $\int \omega > 0$.

algebraic intersection should be euler char b/c of what happens when we wiggle a section.

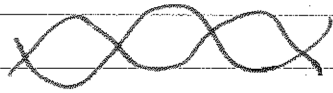
Q: How to classify Lag mflds up to Ham isotopy?
HARD in dim ≥ 2.
→ probably in $\mathbb{R}^4$ by Georgois D-R
→ $\mathbb{CP}^2$ have some ideas
→ $\mathbb{R}^6$ no clue, only a few examples.

Floer cohomology, when defined, $HF^*(L, L')$
→ attempts to measure in a Ham isotopy inv. way how often Lag's need to intersect.

Properties: $HF^*(L, \varphi(L')) \cong HF^*(L, L')$
$HF^*(L, L) \cong H^*(L)$ if no discs of $\int \omega > 0$.

$rk(HF^*(L, L')) \leq |L \pitchfork L'|$.

$CF(L, L')$ is generated by intersection pts.



$\partial^2 = 0$ $\circlearrowright$ $CF(L, L')$ gen'd by intersections.

when defined,
Says Lag can't be displaced off themselves

Downside: CR problems are only easy in dim 1.
What if we want to prove something about a mysterious Lag we can't write down / know about?

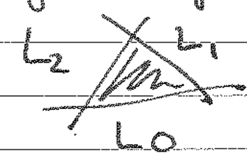
Luckily: don't need to understand all Lag submfld in order to understand their intersection behavior
luckily have Fukaya cats to help.

Fuk cat: $\mathcal{F}(M)$: Obj = Lag submflds (nice ones) w/ extra data

morph: Floer complexes. ($CF^*(L, L')$ which only records their intersection properties.

diff: Floer diff on the morphism spaces.

composition: given by the Floer product



higher compositions — only associative up to htpy.
→ e.g. htpy data for failure of associativity

Get a Fukaya A_∞-category
→ Derived Fukaya category w/ triangulated str

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u] & \end{tikzcd}$$

→ can relate alg. prop of new objects in terms of old objects.

→ often Fukaya cats have generating objects G.

G (split) generates $\mathcal{F}(M)$ if every obj is q-iso to a (direct summand of) iterated mapping cone built from copies of G.

Why care? If we understand the morphisms of G, $A = End(G)$ including A_∞ str. then we actually understand arbitrary objects.

$\mathcal{F}(M) \hookrightarrow$ mod-A fully faithful embedding.
$T \mapsto Hom(G, T)$

w/ nice techniques by Abouzaid-Seidel
Ganatra, Sheridan.

Remark: classifying Lags up to Floer iso $\neq$ Ham iso.

noncompact case: need to be careful "at" ∞.
how do we want to count intersections?
Wrapped Fuk cat
Fukaya-Seidel cat

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MIRROR SYMMETRY ~ 1990
came out of topological string theory.
Physicists were studying CY 3-folds as their model for space time
$X^3_{\mathbb{C}}$, $K_X = \Omega^{3,0} \cong \mathcal{O}_X$ hol. volume form
these seem to come in pairs s.t.

$$H^{p,q}(X) = H^{n-p,q}(X^\vee).$$

$S^2 \to X$
Also the GW inv. counting hol. spheres in X can be predicted from Hodge period $\int$ theory on $X^\vee$.
→ the physicists could do it but not the math peeps! continue. (Givental, Lian-Liu-Yau

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☆ Kontsevich 1994 HMS (what we will focus on)

$$D^{\pi}\mathcal{F}(X) \simeq D^b\mathrm{Coh}(X^\vee).$$

formal summands of bded complexes ↑ ↑ bded complexes

(HH^* computes QH^* ... Ganatra - Perutz - Sheridan)
can use this to make enumerative predictions.

Roughly: Lag submflds of X ⟷ Coh. sheaves on $X^\vee$

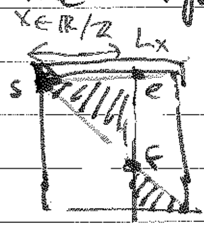
Idea of Coh Sheaves: analytic/algebraic mflds w/ analytic/algebraic VB's over them

Lag $\cap$ (~~count~~ counted by $\#\Delta$) ⟷ Ext ($\cap$, $H^*_{\bar\partial}$)
↑ Dolbeault cohomology

so in the right sense the intersection theories match too.

ex: T^2

interested: Polischuk-Zaslow. (1998)



$$L_0 \xrightarrow[s]{} L_1 \xrightarrow[e]{} L_x$$ (huge Δ.)

$$e \cdot s = \left(\sum_{n\in\mathbb{Z}} T^{(n+x)^2/2}\right) f.$$

The mirror of a torus is a (torus) elliptic curve
$E = \mathbb{C}/\mathbb{Z}+i\tau\mathbb{Z}$.

how many deg 1 sections are there? a 1D space via Riem. Roch

$$\mathcal{O} \xrightarrow{s} \mathcal{L}_1 \xrightarrow{e} \mathcal{O}_x \leftarrow \text{pt.} \quad \rightsquigarrow \; = e^{-\pi i\tau x^2}\sum_{n\in\mathbb{Z}} e^{2\pi i\tau(n+x)^2/2}$$

Jacobi theta functions $\vartheta(\tau, \tau x) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n x\tau}$

next: HMS was subsequently extended to noncompact & non-CY examples.
But these formulas involve new beasts

wrapped, Fukaya-Seidel
matrix fact., sing.

easier in noncompact case
in compact case hard: just elliptic curves
$K3$ w/ a lot of trouble
T^4

one viewpoint:
$$\mathcal{F}(X) \longleftrightarrow \text{mod-}\underset{\underset{\text{End}(G)}{\shortparallel}}{A}$$

If $X^\vee = \operatorname{Spec} R$ then $D^b\operatorname{Coh}(X^\vee) = \text{mod-}R$,
↑ eq. affine

in general, the algebraization of SG ⇒ non-comm alg. geometry!

Remarkable Fact: often this gives an honest (comm) alg. space instead of some horrible noncomm. space.

Q: How to build such a candidate $X^\vee$ geometrically? (beyond just saying its Spec of something).

AG: everything is a moduli space.

$X^\vee$ = moduli space of pts of $X^\vee$

$\mathcal{O}_p$ skyscraper $\xleftrightarrow[\text{HMS}]{} L_p$

$HF^*(L_p, L_p)$

$HF^*(L_p, L_p) \cong \operatorname{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong H^*(T^n)$ good thinking

⇝ $X^\vee$ = moduli space of Lag T^n in $\mathcal{F}(X)$.

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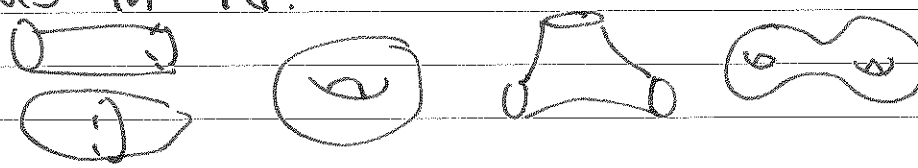
Strominger - Yau - Zaslow

this is consistent w/ SYZ conj of '96

Plan : 1) LF & Fukaya categories
2) HMS in 1d.



3) SYZ philosophy & construction of mirrors
4) stuff in progress / hopes for future.