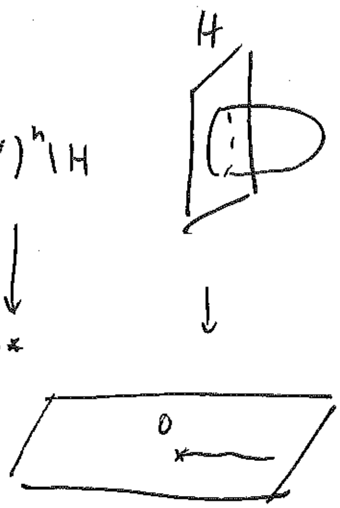
$$\tilde{H} := (\mathbb{C}^*)^n \setminus H = \{x_{n+1} - f(x_1, \dots, x_n) = 0\} \subset (\mathbb{C}^*)^{n+1}$$

$$\Big\updownarrow \text{HMS}$$

$$(\tilde{Y} = \underset{z}{\mathbb{C}} \times Y,\ \tilde{W} = zW) \overset{\text{Orlov}}{\longleftrightarrow} D^b\mathrm{Coh}(Z)$$

$$\begin{array}{c} (\mathbb{C}^*)^n \setminus H \\ f \big\downarrow \\ \mathbb{C}^* \end{array}$$

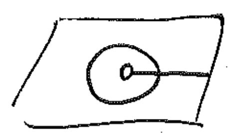


Can find embedding
$(\mathbb{D}^2 \setminus \{0\}) \times H \hookrightarrow (\mathbb{C}^*)^n \setminus H$.
Can arrange to be a Liouville subdomain.
Abouzaid-Seidel restriction:

$$\mathcal{W}((\mathbb{C}^*)^n \setminus H) \longrightarrow \mathcal{W}(\mathbb{C}^* \times H)$$

$$\Big\downarrow \text{"generic fiber"} \quad \text{"residue"}$$

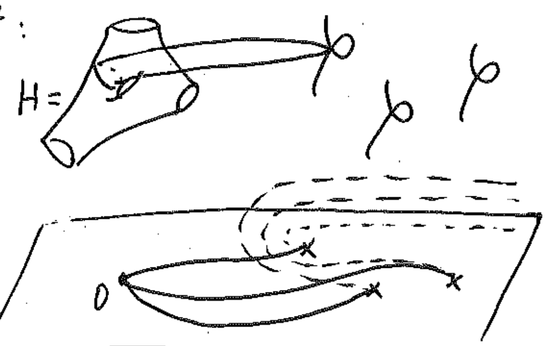
$$\mathcal{W}(H)$$



<u>Ex</u>:

$$\begin{array}{c} (\mathbb{C}^*)^2 \\ \big\downarrow z_1 + z_2 + \frac{1}{z_1 z_2} \\ \mathbb{C} \end{array}$$

mirror to $\mathbb{CP}^2$:



$$\begin{tikzcd}
FS\big((\mathbb{C}^*)^2, \underbrace{z_1 + z_2 + \tfrac{1}{z_1 z_2}}_{f}\big) \arrow[r, leftrightarrow] \arrow[d, "\text{remove fiber } H"'] & D^b\mathrm{Coh}(\mathbb{P}^2) \arrow[d, "i_*"] \arrow[dr] & \\
\mathcal{W}((\mathbb{C}^*)^2 \setminus H) \arrow[r, leftrightarrow] \arrow[d, "\text{residue}"'] & D^b\mathrm{Coh}(Z) \arrow[d] & \mathrm{Perf}(Z) \\
\mathcal{W}(H) \arrow[r, leftrightarrow] \arrow[rr, bend right, "\text{classical HMS}"'] & D^b_{sing}(Z) \underset{\text{stab}}{\overset{\text{Orlov}}{\simeq}} & D^b_{coh}(\{x_0 x_1 x_2 = 0\})
\end{tikzcd}$$

- from going to ∞ along dotted curves

$\mathrm{Perf}(Z)$ (dies in D^b_{sing})

nodal elliptic curve

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Joint work in progress w/ Abouzaid:

HMS for $H = \begin{cases} \text{pants, higher dim'l pants} & \Pi_n = (\mathbb{C}^*)^n \setminus \{\sum x_i + 1 = 0\} \\ \text{hypersurfaces in } (\mathbb{C}^*)^n & \cong \{\sum x_i + 1 = 0\} \subset (\mathbb{C}^*)^{n+1} \\ \text{hypersurfaces in toric varieties} & \\ \text{complete intersections} & \end{cases}$

Want: $D^b\mathrm{Coh}(H)$ vs Fukaya categ of mirror.

Geometric setup: (Abouzaid - A - Katzarkov)

$$H = \Big\{ f_t = \sum_{\substack{\alpha \in A \subseteq \mathbb{Z}^n \\ \text{finite}}} c_\alpha t^{\rho(\alpha)} x^\alpha = 0 \Big\} \subset (\mathbb{C}^*)^n$$

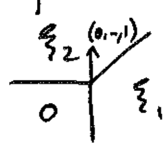
generalized SYZ mirror $\updownarrow$ $\quad \rho : A \to \mathbb{R}$ convexity; $\quad |t| \ll 1$ Novikov parameter

(Y, W), $Y =$ toric variety (CY), $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta \geq \mathrm{Trop}\, f(\xi)\}$

$W = -z^{(0,\dots,0,1)}$ $\qquad \mathrm{Trop}\, f(\xi) = \max_{\alpha \in A} (\langle \alpha, \xi \rangle - \rho(\alpha))$

Eg: Pants $\quad H = \{1 + x_1 + \dots + x_n = 0\}$

$\mathrm{Trop}\, f = \max(0, \xi_1, \dots, \xi_n) \longleftrightarrow Y = \mathbb{C}^{n+1}, \quad W = -\prod z_i$



"Generalized SYZ"

$H \underset{\text{Fuk categ } D^b}{\overset{\text{equivalent}}{\longleftrightarrow}} (X, W_X)$, W_X Morse-Bott w/ $\mathrm{Crit}\, W_X \simeq H$

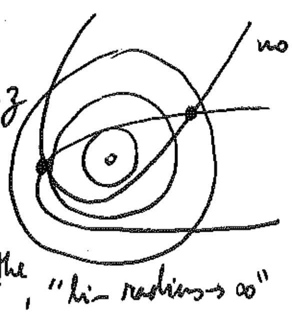
$X = Bl_{H \times 0} ((\mathbb{C}^*)^n \times \underset{y}{\mathbb{C}})$, $W_X = \pi^*(y)$

$X \supset X^0$ open dense carrying a Lagr T^{n+1}-fibration.

Build $Y^\circ \subset Y$ as moduli space of T^{n+1}-objects in $\mathcal{F}(X^\circ)$.

* $\mathcal{F}(X^\circ, W_X)$ adds a few extra T^{n+1}-objects $\rightsquigarrow$ get Y instead of Y°

$\mathbb{C}^*_z, W=z$



mirror extends (add in Lagrs over these curves $\subset$ as "generalized tori")

$\longleftrightarrow$ origin in $\mathbb{K} \supset \mathbb{K}^*$

adding this W introduces the new object, "Lin radius $\to \infty$"

* Taking X instead of X°: Lagr T^{n+1} become weakly unobstructed.

  Count of Maslov 2 discs $\rightsquigarrow W \in \mathcal{O}(Y)$

  └ from $\mathbb{C}_y$ factor in X in previous page

$D^b\mathrm{Coh}(H) \simeq \mathcal{F}(Y, W)$? $\quad H \subset (\mathbb{C}^*)^n$ hypersurface

1- Can define $\mathcal{F}(Y,W)$, if (Y,W) toric

2- Can construct an object of $\mathcal{F}(Y,W)$ L_0 that is candidate for mirror of $\mathcal{O}_H$ (which generates $D^b\mathrm{Coh}(H)$, since H is affine hypersurf)

3- Calculate $\mathrm{End}(L_0) \underset{A_\infty\text{-alg}}{\overset{\text{as}}{\simeq}} \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \big/ \left(\sum c_\alpha t^{\rho(\alpha)} x^\alpha\right) \simeq \mathrm{End}(\mathcal{O}_H)$

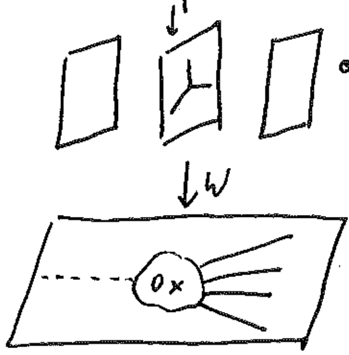
└ Novikov parameter in $\mathbb{K}$.

4- L_0 generates?

<u>Geometry</u>

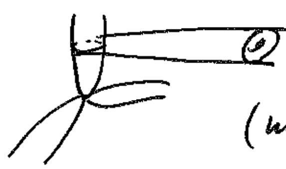
Eg: $\mathbb{C}^{n+1} \xrightarrow{-\prod z_i} \mathbb{C}$

$W^{-1}(0)$ = union of all toric divisors



Toric variety w/ monomial vanishing to deg 1 on every toric divisor

Objs = properly embedded $L \subset Y$ st
$W(L) \subset \mathbb{C}$ outside of compact set, union of radial straight lines $e^{i\theta}\mathbb{R}_+$, $-\pi<\theta<\pi$

Aganagic–Vafa branes:  $S^1\times\mathbb{R}^2 \leftrightarrow$ skyscrapers of pts
(mostly unobstructed, else need bounding cochains)

- Perturb in horizontal direction:
  let ρ^t = flow in $\mathbb{C}$: • id in compact subset
  • at ∞, maps radial $\to$ radial counterclockwise, w/o crossing $\mathbb{R}_{<0}$.

Lift via symplectic parallel transport $\rightsquigarrow \rho_t(L)$ Lagrangian

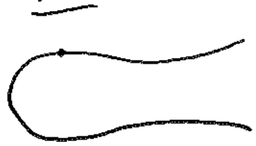
replace  w/  in base

- Wrap in vertical direction: H Hamilt invariant under parallel transport
  fiberwise proper, w/ linear growth
  $\rightsquigarrow$ flow φ^t preserves fibers & commutes w/ ρ_t.

Combine to get
$L^t := \varphi^t \rho^t L$

L "fiberwise admissible at ∞":
 $(\mathbb{C}^*)^n = \{\prod z_i = c\} \subset (\mathbb{C}^*)^{n+1}$.

Whenever $|z_i| \to \infty$ along L
in a fiber of W,
$\arg(z_i) \equiv \theta_i$ fixed eventually



This gives you a max principle, and well-def HF^*.

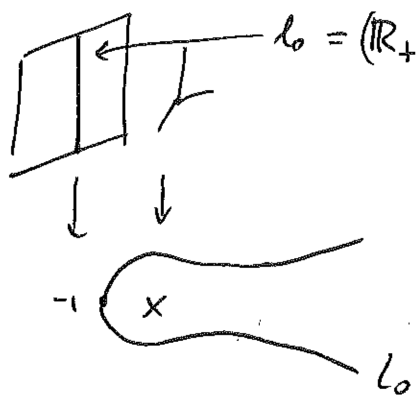
$\hom(L, L') = \varinjlim_{t \to \infty} CF^*(L^t, L')$

Continuation maps : $L \xleftarrow{\text{"id"}} L^{\varepsilon} \xleftarrow{\text{"id"}} L^{2\varepsilon} \leftarrow \cdots$ (Abouzaid-Seidel)

Ideally, would take parallel Transp of

which would be a very singular object, hard to do HF^* for

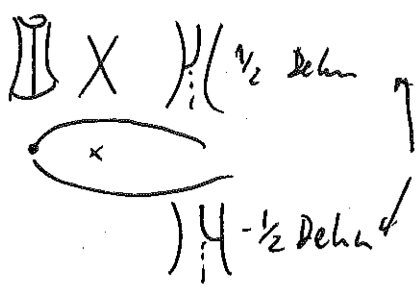
Instead, will consider another L_0 :



(would not close up if took closed loop around origin)

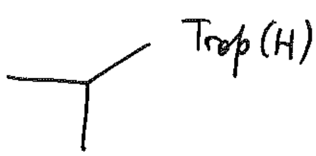
<u>Claim</u> : $\exists$ toric Kähler form on Y st $\arg(z_i)$ condition preserved by parallel transport.

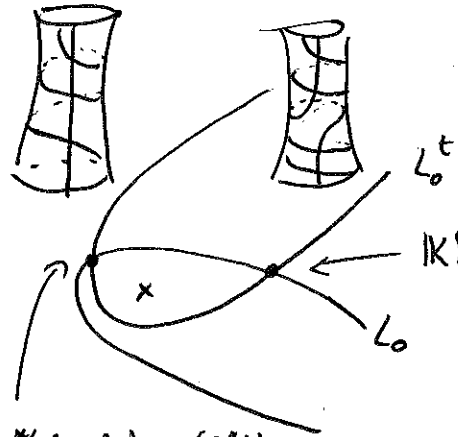
Idea : $\mathbb{C}^2 \xrightarrow{-z_1 z_2} \mathbb{C}$



difference is a full Dehn twist

$(\mathbb{C}^*)^n \xrightarrow{\mu \text{ or } \log} \mathbb{R}^n$





$\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\,\theta$
deg -1

$CW^*(L_0, L_0)$ in $(\mathbb{C}^*)^n$
$\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\,e$
deg 0

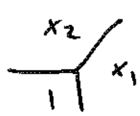
∂ counts holo sections

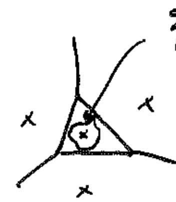
Product: take a third copy

Free module of rank 1 : $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes \mathbb{K}\langle \theta \rangle / \theta^2$ odd

$$\partial(\theta) = \Big(\sum_{\alpha \in A} \kappa_\alpha \, t^{\rho(\alpha)} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \Big)\, e$$

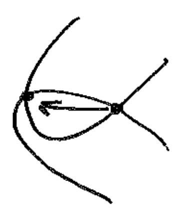
$1 + \dots$ involving local GW $g=0$ of divisors :

(

no GW contribution but if have cpct components 

spheres like this might contribute to nodal disks)

$$H^* \mathrm{Hom}(L_0, L_0) = \lim HF^*(L_0^{t \to \infty}, L_0) \simeq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(f) = \mathcal{O}(H)$$

$\mathcal{O}_X(-H) \xrightarrow{f} \mathcal{O}_X$ from 

Compactify (partially):

$$\begin{array}{ccc} H \subset (\mathbb{C}^*)^n & & \text{pants} \subset (\mathbb{C}^*)^2 \\ \downarrow & & \downarrow \\ \bar{H} \subset V \ \text{toric} & & \mathbb{CP}^1 \subset \mathbb{CP}^2 \end{array}$$

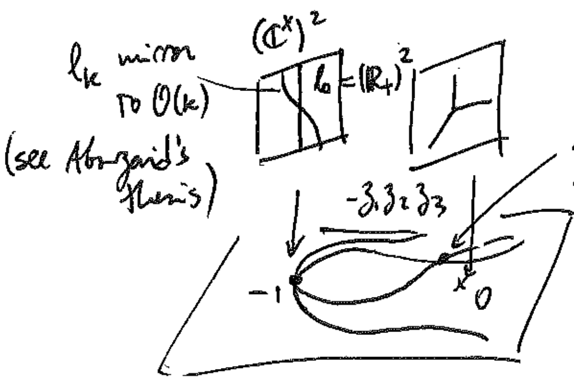
(Fano semi-pos) variety

Mirror: space still Y, $W = -z^{(0,\dots,0,1)} + $ (1 term for each ray of fan)

Eg:  $\longleftrightarrow$ $\mathbb{C}^3$, $-z_1 z_2 z_3$

 $\longleftrightarrow$ $-z_1 z_2 z_3 + q z_1 + q z_2 + q z_3$

Want: Lagrs admissible for each of $-z_1 z_2 z_3, z_1, z_2, z_3$.



$z_1 z_2 z_3 = 1$

$$W_{aux} = q z_1 + q z_2 + q z_3 = q\underbrace{\left(z_1 + z_2 + \frac{1}{z_1 z_2}\right)}_{\text{mirror to } V = \mathbb{CP}^2}$$

hom(ℓ_0, ℓ_k) ~ homog. coords of $\mathbb{CP}^2 = V$

To compute $\mathrm{Hom}(\mathcal{O}|_{\bar{H}}, \mathcal{O}(k)|_{\bar{H}})$, use resolution by

$$\mathrm{Hom}(\mathcal{O}_V, \mathcal{O}_V(k)) \longleftarrow \mathrm{Hom}(\mathcal{O}_V(H), \mathcal{O}_V(k))$$

multiplication by defining section of H.

Similar story for complete intersections