"LG stabilization"

Thm (Orlov): $D^b_{sing}(\mathbb{C}^3, -xyz) \simeq D^b Coh(\{xy=0\})$

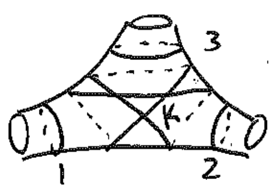
Knörrer periodicity: $(\underset{x,y}{Y} \times \underset{z}{\mathbb{C}}, W = z\underset{\mathcal{O}(Y)}{f} + g) \leftrightarrow (f^{-1}(0) \subset Y, g|_{f^{-1}(0)})$

Symplectic analogue: if $f^{-1}(0)$ smooth, then (in affine setting)
, not in our case...

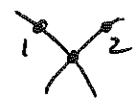
$FS(Y \times \mathbb{C}, W = zf) \leftrightarrow \mathcal{F}(f^{-1}(0) \subset Y)$
$(W?)$

Morse-Bott, crit $W \simeq f^{-1}(0)$

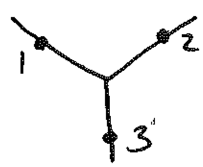
Compact Lags:



$\leftrightarrow$ skyscrapers of points



$\updownarrow$ stabil



crit $W = \bigcup$ 3 coord axes

(3 is not $\mathbb{Z}$-graded ...)

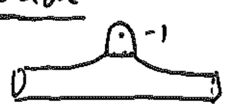
$K \underset{\text{Seidel}}{\longleftrightarrow} \mathcal{O}_{0 \in \mathbb{C}^3}$ in D^b_{sing}

$[\mathcal{O}_{pt}]$ in D^b_{sing}
$= A \oplus A[1]$

$0 \to \mathcal{O}_A \oplus \mathcal{O}_B \to \mathcal{O} \to \mathcal{O}_0 \to 0$

Compactification

(*) think of T as deformation parameter (Seidel, Sheridan) assoc. to divisor $\{-1\}$

$\mathbb{C}^* =$  $=$  $\cup \{-1\} \leftrightarrow (\mathbb{C}^3, \underbrace{-xyz + Tz}_{(T-xy)z})$

$(T-xy)z$, has MB sing along $\{xy=T\} \subset \mathbb{C}^2_{12}$
$\simeq \mathbb{C}^*$

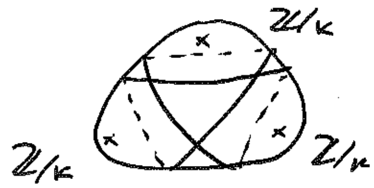
Similarly,

$\mathbb{CP}^1 =$  $\cup \{3 pts\} \leftrightarrow (\mathbb{C}^3, -xyz + Tx + Ty + Tz)$

$\mathbb{K}^*$, if think of T as Novikov parameter (*)

Thm (Orlov): $D^b_{sing}(\mathbb{C}^3, -xyz + Tx + Ty + Tz) \simeq D^b_{sing}(\{xy=T\}, \underline{Tx+Ty}) \simeq$
$\simeq (\mathbb{C}^*, z + \frac{T^3}{z})$

Orbifold compactification:



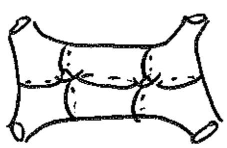
$\xleftrightarrow[\text{(Efimov)}]{\text{Seidel}}$ $(\mathbb{C}^3, -xyz + T(x^k + y^k + z^k))$

$k = 2g+1$ $\quad \uparrow k:1$

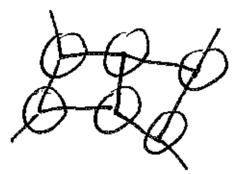
Σ_g  $\longleftrightarrow (\dots)/\mathbb{Z}/k$

This gives mirror symmetry for Σ_g.

More generally: given a pair-of-pants decomposition



$\longleftrightarrow$ LG model w/ charts $(\mathbb{C}^3, -xyz)$



Čech model

$$W(-) \leftrightarrow D^b_{sing}$$

See Heather Lee's thesis.

<u>Next</u>: Higher dimensions.

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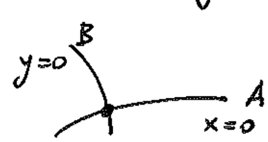
<u>Recall</u>

$X =$ 

$\longleftrightarrow \quad X^\vee = \{xy=0\} \subset \mathbb{A}^2$



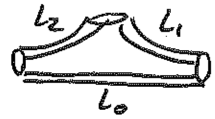
$$HW^*(L_0, L_0) = \mathbb{K}[x,y]/xy=0 = \mathrm{End}(\mathcal{O}_{X^\vee})$$

L_0 doesn't generate

L_1, L_2 mirror to $\mathcal{O}_A$, $\mathcal{O}_B$

$$\mathrm{Ext}^*(\mathcal{O}_A, \mathcal{O}_A) = \mathbb{K}[y,z]/yz$$

$\deg z = 2$

Have $D^b\mathcal{W}\Big($  $\Big) \simeq D^b\mathrm{Coh}(X^\vee)$

$$\begin{array}{ccll} L_0 & & \mathcal{O} & (xy=0) \\ L_1 & \longleftrightarrow & \mathcal{O}_A & (x=0) \\ L_2 & & \mathcal{O}_B & (y=0) \end{array}$$

A_∞-strs also match: on left, have two exact triangles, coming from shaded triangle in front and its counterpart in back. These triangles correspond to

$$\begin{tikzcd} \mathcal{O} \arrow[r, shift left] \arrow[dr, shift left] & \mathcal{O}_A \arrow[l, shift left] \arrow[d, shift left] \\ & \mathcal{O}_B \arrow[ul, shift left] \arrow[u, shift left] \end{tikzcd}$$

.

There's also <u>stabilized LG mirror</u>: (useful next week, for other direction of HMS)

$$D^b\mathrm{Coh}(X^\vee) \simeq D^b_{sing}(\mathbb{C}^3, -xyz) = \mathrm{Coh}(\{xyz=0\})/\mathrm{Perf}$$

Analogue of $\mathcal{O}_{\{xy=0\}}$ is now $\mathcal{O}_{\{z=0\}}[1]$.

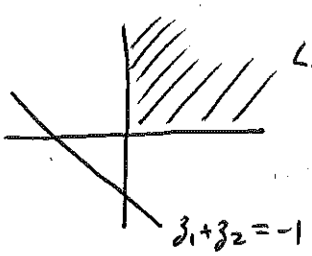
$\mathcal{O}_{\{x=0\}}$ & $\mathcal{O}_{\{y=0\}}$ are as before

<u>Higher dimensional pairs of pants</u>

(building blocks of alg subvarieties in toric varieties in tropical geometry) locally $(\mathbb{C}^*)^n$

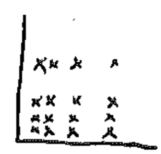
$$\Pi_n = \mathbb{CP}^n \setminus \{n+2 \text{ hyperplanes}\} \quad (\text{generic})$$
$$= (\mathbb{C}^*)^n \setminus \{z_1 + \dots + z_n = -1\}$$

$n=2$:



∞ sympl volume near each puncture, (completed near removed axes)

generators for $CW^*(L_0, L_0)$:



$\mathbb{N}^2$-many near corner

$$HW^*_{(\mathbb{C}^*)^n}(\mathbb{R}^n_+, \mathbb{R}^n_+) \cong \mathbb{K}[x_1^\pm, \dots, x_n^\pm] \cong \mathbb{K}[x_1, \dots, x_{n+1}]/_{x_1\cdots x_{n+1}=1}$$

one generator wrapping around each hyperplane

x_{n+1}: hyperplane at ∞

In Π_n, $HW^*_{\Pi_n}(\mathbb{R}^n_+, \mathbb{R}^n_+) \cong \mathbb{K}[x_1, \dots, x_{n+1}]/_{x_1\cdots x_{n+1}=0}$

(removing one hyperplane, fewer triangles)

So, candidate $X_n^\vee = \{x_1 \cdots x_{n+1} = 0\} \subset \mathbb{A}^{n+1}$

$$\begin{array}{c} X_n^\vee \\ \text{stabilization} \updownarrow \text{(Orlov)} \\ \text{LG model } \ (\mathbb{A}^{n+2}, W = -x_1 \cdots x_{n+2}) \end{array}$$

(Orlov periodicity is instance of Knörrer periodicity (Lagr multipls...) : if $Y = f^{-1}(0) \subset Z$, $f, g \in \mathcal{O}(Z)$, then $(Z \times \mathbb{C}, fz + g) \leftrightarrow (Y, g|_Y)$)

$\mathcal{W}(\Pi_n)$ not yet calculated for $n \geq 2$, but

- Sheridan: compact immersed Lagr $S^n \subset \Pi_n \leftrightarrow \mathcal{O}_{\{0\}}$ in LG mirror
- Nadler: microlocal sheaves calc of $\mu Sh^{W}(\Pi_n)$
  ($\mathcal{W} \leftrightarrow$ by Ganatra - Pardon - Shende)

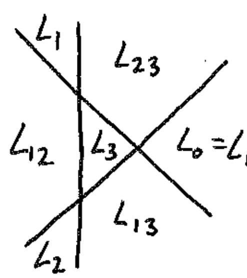
Consecutive dimensions fit together:

$$\begin{array}{ccc} \mathcal{W}(\Pi_{n+1}) & \xleftrightarrow[\text{conj}]{\text{HMS}} & D^b Coh(X_{n+1}^\vee) \\ \boxed{?} \Big\downarrow & & \Big\downarrow \text{quotient} \\ \mathcal{W}(\Pi_n) \xleftrightarrow[\text{conj}]{\text{HMS}} & D^b_{sing}(\mathbb{C}^{n+2}, -x_1 \cdots x_{n+2}) & = D^b Coh(X_{n+1}^\vee)/\langle \mathcal{O} \rangle \end{array}$$

<u>Q</u>: What ~~do~~ is the appropriate quotient map $\mathcal{W}(\Pi_{n+1}) \to \mathcal{W}(\Pi_n)$?

$\Pi_n = (\mathbb{C}^*)^n \setminus \Pi_{n-1}$

$$\begin{array}{c} \Pi_n \\ \Big\downarrow z_1 + \cdots + z_n + 1 \\ \mathbb{C}^* \end{array}$$



$L_0 = L_{123} = (\mathbb{R}_+)^n \longleftrightarrow \mathcal{O}$ (corresponds to union of 3 hyperplanes)

$\downarrow$

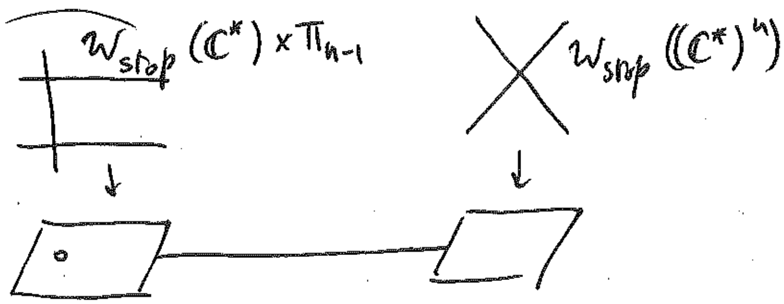
$L_{I \subset \{1,\dots,n+1\}} \longleftrightarrow \mathcal{O}_{\underbrace{\{\prod_{i \in I} x_i = 0\}}_{H_I}}$

$\Pi_2 \longleftrightarrow \{x_1 x_2 x_3 = 0\} \subset \mathbb{A}^3$

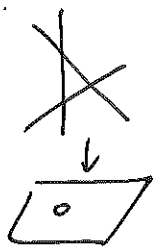
$\mathcal{O}_{H_I}, \mathcal{O}_{H_J}, \mathcal{O}_{H_{I \cup J}}$

2. Sylvan: Think of previous picture as obtained from gluing two simpler ones: (in progress)

partially wrapped



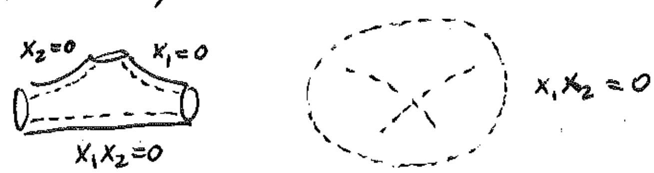
There's a pushout diagram building $\mathcal{W}(\Pi_n)$



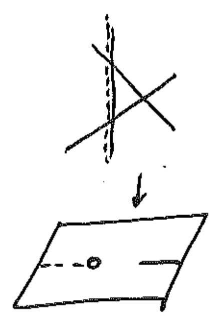
More generally, can "glue partially wrapped categories along stops."

In low dim, have

1-dim'l pair-of-pants



Product w/ $\mathbb{A}^1_{x_3}$ gives



"pull back & include"

This is the conjectured origin of the map $\boxed{?}$ in diagram of previous page.
It corresponds to restricting to missing fiber ($L_0 = L_{123}$ dies)

We'll now explain a more general conjectural picture:

Conjectural picture of HMS for hypersurfaces in $(\mathbb{C}^*)^n$:

$$H = \{ f_t(z_1,\dots,z_n) = \sum_{\alpha\in A\subset\mathbb{Z}^n} c_\alpha t^{\rho(\alpha)} z^\alpha = 0 \} \subset (\mathbb{C}^*)^n$$

Laurent poly

Tropical geometry $\downarrow$ $t\to 0$ scaling limit

$\rho : A \to \mathbb{R}$ convex wrt some max. subd. of $\mathrm{Conv}(A)$

$t\to 0$ parameter $\log(H)$ is amoeba converging to tropical hypersurf

$$\varphi = \mathrm{Trop}\, f : \mathbb{R}^n \to \mathbb{R}\,, \quad \varphi(\xi_1,\dots,\xi_n) = \max_{\alpha\in A}\big(\langle\alpha,\xi\rangle - \rho(\alpha)\big).$$

stabilize first (∃ a few non-cpt objects behaving like tori, also singular tori...)

From (slightly generalized) SYZ perspective, a candidate mirror is: (AAZ)

$H^{n-1} \overset{\text{mirror}}{\longleftrightarrow}$ toric LG model (Y, W)

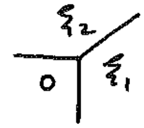
Y toric $(n+1)$-fold w/ moment polytope

$$\Delta_Y = \{ (\xi_1,\dots,\xi_n,\eta) \in \mathbb{R}^n\oplus\mathbb{R} \mid \eta \geq \mathrm{Trop}\, f(\xi_1,\dots,\xi_n) \}$$

$W = -z^{(0,\dots,0,1)}$ toric monomial vanishing to order 1 on all toric divisors

Ex 1: $f(z_1,\dots,z_n) = 1 + z_1 + \dots + z_n \quad\longleftrightarrow\quad \varphi = \max(0, \xi_1, \dots, \xi_n)$

$H = \Pi_{n-1} \subset (\mathbb{C}^*)^n$

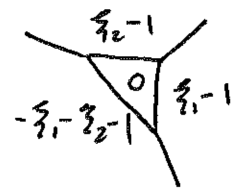


$Y = \mathbb{C}^{n+1}$

$W = -x_1\cdots x_{n+1}$

Ex 2: $f(z_1,z_2) = t\left(z_1 + z_2 + \frac{1}{z_1 z_2}\right) + 1 \quad\longleftrightarrow$

(∃ higher dim version



$Y = \{\mathcal{O}(-3) \to \mathbb{P}^2\}$ Total space of (x_3; $x_0 x_1 x_2$)

$W = -x_0 x_1 x_2 x_3$

Expect: ($\boxed{?}$ above?)

$$\begin{array}{ccc} \mathcal{W}(H) & \overset{HMS}{\longleftrightarrow} & D^b_{sing}(W^{-1}(0)) = D^b\mathrm{Coh}(Z)/\mathrm{Perf}\,, \quad Z = W^{-1}(0) \subset Y \\ \uparrow \rho & & \uparrow \text{quotient} \qquad\qquad\qquad\qquad\qquad = \cup \text{ toric divisors} \\ \mathcal{W}((\mathbb{C}^*)^n \setminus H) & \overset{HMS}{\longleftrightarrow} & D^b\mathrm{Coh}(Z) \end{array}$$

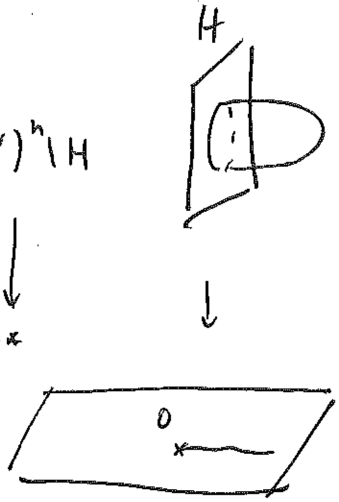
ρ: "restriction to missing fiber". (Andrei: "residue" ...)

$$\tilde{H} := (\mathbb{C}^*)^n \setminus H = \{x_{n+1} - f(x_1, \dots, x_n) = 0\} \subset (\mathbb{C}^*)^{n+1}$$

$$\updownarrow \text{HMS}$$

$$(\tilde{Y} = \underset{z}{\mathbb{C}} \times Y,\ \tilde{W} = zW) \overset{\text{Orlov}}{\longleftrightarrow} D^b\mathrm{Coh}(Z)$$

$$(\mathbb{C}^*)^n \setminus H \xrightarrow{f} \mathbb{C}^*$$



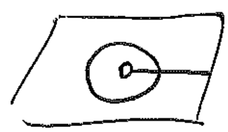
Can find embedding

$(\mathbb{D}^2 \setminus \{0\}) \times H \hookrightarrow (\mathbb{C}^*)^n \setminus H$.

Can arrange to be a Liouville subdomain.

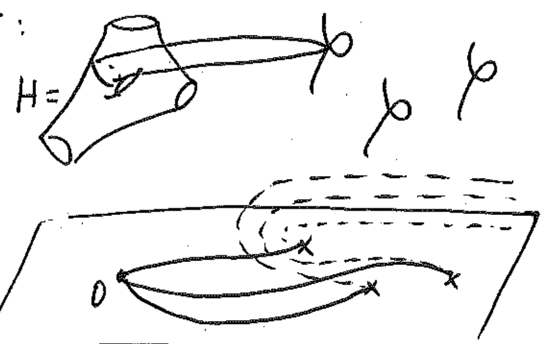
Abouzaid-Seidel restriction:

$$\mathcal{W}((\mathbb{C}^*)^n \setminus H) \longrightarrow \mathcal{W}(\mathbb{C}^* \times H)$$



$$\mathcal{W}(\mathbb{C}^* \times H) \xrightarrow{\text{"generic fiber"},\ \text{"residue"}} \mathcal{W}(H)$$

<u>Ex</u>: $(\mathbb{C}^*)^2 \xrightarrow{z_1 + z_2 + \frac{1}{z_1 z_2}} \mathbb{C}$ mirror to $\mathbb{CP}^2$:



$$\begin{tikzcd}
FS\big((\mathbb{C}^*)^2, \underbrace{z_1 + z_2 + \frac{1}{z_1 z_2}}_{f}\big) \arrow[r, leftrightarrow] \arrow[d, "\text{remove fiber } H"'] & D^b\mathrm{Coh}(\mathbb{P}^2) \arrow[d, "i_*"] \arrow[dr] & \\
\mathcal{W}((\mathbb{C}^*)^2 \setminus H) \arrow[r, leftrightarrow] \arrow[d, "\text{residue}"'] & D^b\mathrm{Coh}(Z) \arrow[d] & \mathrm{Perf}(Z) \\
\mathcal{W}(H) \arrow[r, leftrightarrow] \arrow[rr, bend right, "\text{classical HMS}"'] & D^b_{sing}(Z) \underset{\text{stab}}{\overset{\text{Orlov}}{\simeq}} & D^b_{coh}(\{x_0 x_1 x_2 = 0\})
\end{tikzcd}$$

– from going to ∞ along dotted curves

$\mathrm{Perf}(Z)$ (dies in D^b_{sing})

nodal elliptic curve

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Joint work in progress w/ Abouzaid:

HMS for $H = \begin{cases} \text{pants, higher dim'l pants} & \Pi_n = (\mathbb{C}^*)^n \setminus \{\sum x_i + 1 = 0\} \\ \text{hypersurfaces in } (\mathbb{C}^*)^n & \quad \cong \{\sum x_i + 1 = 0\} \subset (\mathbb{C}^*)^{n+1} \\ \text{hypersurfaces in toric varieties} & \\ \text{complete intersections} & \end{cases}$

Want: $D^b\mathrm{Coh}(H)$ vs Fukaya categ of mirror.

Geometric setup: (Abouzaid - A - Katzarkov)

$$H = \left\{ f_t = \sum_{\substack{\alpha \in A \subseteq \mathbb{Z}^n \\ \text{finite}}} c_\alpha t^{\rho(\alpha)} x^\alpha = 0 \right\} \subset (\mathbb{C}^*)^n$$

generalized SYZ mirror $\updownarrow$

$\rho : A \to \mathbb{R}$ convexity

$|t| \ll 1$ Novikov parameter

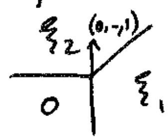
(Y, W), $Y =$ toric variety (CY), $\Delta_Y = \{ (\xi, \eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta \geq \mathrm{Trop}\, f(\xi) \}$

$W = -z^{(0,\dots,0,1)}$ $\qquad \mathrm{Trop}\, f(\xi) = \max_{\alpha \in A} \left(\langle \alpha, \xi \rangle - \rho(\alpha) \right)$

Eg: Pants $\quad H = \{ 1 + x_1 + \dots + x_n = 0 \}$

$\mathrm{Trop}\, f = \max(0, \xi_1, \dots, \xi_n) \longleftrightarrow Y = \mathbb{C}^{n+1}$

$W = -\prod z_i$



"Generalized SYZ"

$H \underset{\text{Fuk categ } D^b}{\overset{\text{equivalent}}{\longleftrightarrow}} (X, W_X)$, W_X Morse-Bott w/ $\mathrm{Crit}\, W_X \cong H$

$X = \mathrm{Bl}_{H \times 0} \left((\mathbb{C}^*)^n \times \underset{y}{\mathbb{C}} \right)$, $W_X = \pi^*(y)$

$X \supset X^0$ open dense carrying a Lagr T^{n+1}-fibration.