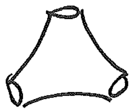
11/17

(so far: compactified $\mathbb{C}^*$ to Fano $\mathbb{CP}^1$
today: go "the other way", from $\mathbb{C}^*$ to pair-of-pants)

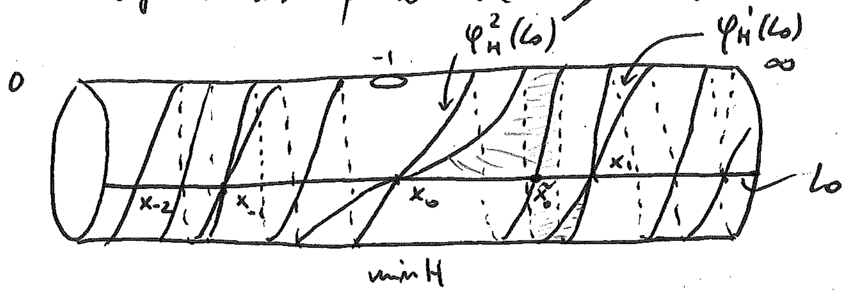
The pair of pants

$X =$  $= \mathbb{P}^1 \setminus \{3\text{ pts}\} = \mathbb{C}^* \setminus \{-1\}$

Want: Y st $\mathcal{W}(X) \simeq D^b Coh(Y)$.

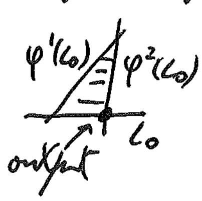
Compute: $CW(L_0, L_0)$, where L_0 is 

Think of X as punctured cylinder



$CW^*(L_0, L_0) = \text{Span}\{x_i, \ i \in \mathbb{Z}\}$

In , $x_i \cdot x_j = x_{i+j} \quad \forall i,j \in \mathbb{Z}$

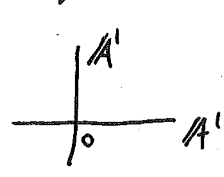
Product counts  triangles that do **not** pass through -1

Calculation: discarding disks through -1,

$$x_i \cdot x_j = \begin{cases} x_{i+j}, & \text{if } ij \geq 0 \\ 0, & \text{if } ij < 0 \end{cases}$$

Call $x_1 =: x$, $x_2 =: y$ and get

$$CW^*(L_0, L_0) \simeq \mathbb{K}[x,y]/xy \quad \text{as } A_\infty\text{-algs}$$

So, **guess** $Y = \text{Spec}\, \mathbb{K}[x,y]/xy =$ 

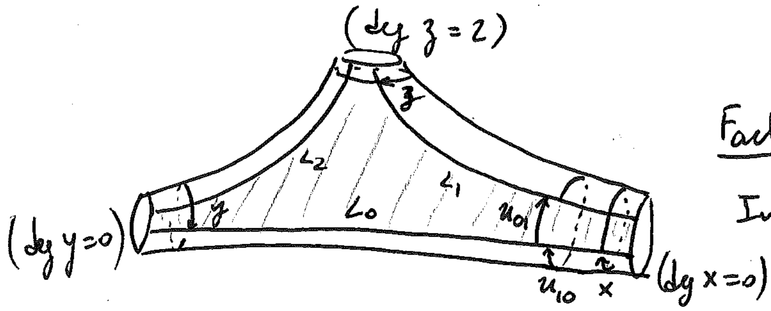
This turns out to be correct.

But... unlike case of cylinder,

L_0 does <u>not</u> split-gen $\mathcal{W}(\ )$

$\mathcal{O}$ does not split-gen $D^b\mathrm{Coh}(+)$

Yet, $\left.\begin{array}{l}\mathcal{W}(\ ) \\ D^b(+)\end{array}\right\} \hookrightarrow \text{mod-}\left(\mathbb{K}[x,y]/xy\right)$, but only hit <u>perfect</u> complexes in $D^b\mathrm{Coh}(+)$



<u>Fact</u>: L_0, L_1, L_2 generate $\mathcal{W}(X)$.
In fact, 2 of them generate.

Morphisms:

$$\begin{tikzcd}
 & L_0 \arrow[loop above, "{\mathbb{K}[x,y]/xy}"] \arrow[dl, shift left, "{u_{10}\,\mathbb{K}[x]}"] \arrow[dr, shift left, "{u_{20}\,\mathbb{K}[y]}"] & \\
L_1 \arrow[loop left, "{\mathbb{K}[x,z]/xz}"] \arrow[ur, shift left, "{u_{01}\,\mathbb{K}[x]}"] \arrow[rr, shift right, "{u_{12}\,\mathbb{K}[z]}"'] & & L_2 \arrow[loop right, "{\mathbb{K}[y,z]/xz}"] \arrow[ul, shift left, "{u_{02}\,\mathbb{K}[y]}"] \arrow[ll, shift right, "{u_{21}\,\mathbb{K}[z]}"']
\end{tikzcd}$$

$u_{01}\,u_{10} = x$

$u_{01}\,u_{12} = 0$

Triangle in front gives

$\mu^3(u_{20}, u_{12}, u_{01}) = \pm\, id_{L_0}$ and cyclic permutations.

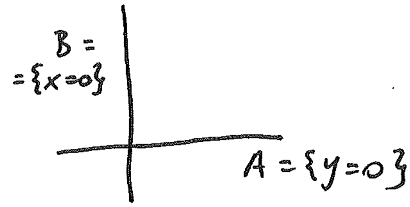
Triangle in back gives

$\mu^3(u_{10}, u_{21}, u_{02}) = \pm\, id_{L_0}$ and cyclic perms..

<u>Prop</u>: (AAEKO): These operations determine the A_∞-str on $\mathcal{A} = \mathrm{End}(L_0 \oplus L_1 \oplus L_2)$.

<u>Note</u>: putting more punctures in sphere increases order of non-trivial higher product μ^k.

Mirror:



$\mathcal{O}_Y, \mathcal{O}_A, \mathcal{O}_B$ generate. Y, A, B are specs of

$R = \mathbb{K}[x,y]/xy$, $R/y = \mathbb{K}[x]$, $R/x = \mathbb{K}[y]$, resp.

$\exists\ z \in \mathrm{Ext}^2(R/y, R/y)$:

Proj resolution: $\cdots \to R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \longrightarrow R/y \to 0$

$\leadsto$ hom$(\cdot, R/y)$ $\quad R/y \xrightarrow{y=0} R/y \xrightarrow{x} R/y \xrightarrow{y=0} R/y \to R/y \to \cdots$

coho: $\quad R/y \qquad \mathbb{K}\ z \qquad \mathbb{K}\ z^2 \qquad \cdots$

Degrees are correct: $\deg z = 2$ matches degree in HF

Get

$$\begin{tikzcd}
 & \mathcal{O}_Y \arrow[loop above, "{\mathbb{K}[x,y]/xy}"] \arrow[dl, shift right, "{\mathbb{K}[x]}"'] \arrow[dr, shift left, "{y\mathbb{K}[y]}"] & \\
\mathcal{O}_A \arrow[ur, shift right, "{x\mathbb{K}[x]}"'] \arrow[loop left, "{\mathbb{K}[x,z]/xz}"] \arrow[rr, shift left, "{u\mathbb{K}[z],\ v\mathbb{K}[z]}"] & & \mathcal{O}_B \arrow[ul, shift left, "{\mathbb{K}[y]}"] \arrow[ll, shift left] \arrow[loop right, "{\mathbb{K}[y,z]/yz}"]
\end{tikzcd}$$

$\deg z = 2$

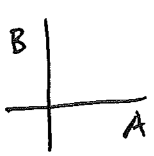
Note: In sympl side, L_0, L_1, L_2 seem interchangeable. But $\mathcal{O}_Y, \mathcal{O}_A, \mathcal{O}_B$ don't seem. There is an <u>ungraded underlying symmetry</u> in the mirror. More below.

μ^3: $\exists$ exact triangles

$$\begin{tikzcd}
L_0 \arrow[rr, "u_{01}"] & & L_1 \arrow[dl, "u_{12}"] \\
 & L_2 \arrow[ul, "u_{20}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
L_1 \arrow[rr, "u_{10}"] & & L_0 \arrow[dl, "u_{02}"] \\
 & L_2 \arrow[ul, "u_{21}"] &
\end{tikzcd}$$

$$L_2 \simeq \mathrm{Cone}(L_0 \xrightarrow{u_{01}} L_1) \simeq \mathrm{Cone}(L_1 \xrightarrow{u_{10}} L_0)$$

In mirror:



have SESs $\quad 0 \to \mathcal{O}_A \to \mathcal{O} \to \mathcal{O}_B \to 0$

$(0 \to R/y \to R \to R/x \to 0)$

and $\quad 0 \to \mathcal{O}_B \to \mathcal{O} \to \mathcal{O}_A \to 0$

<u>Thm</u>: (AAEKO): $D^b\mathrm{Coh}(X) \simeq \mathcal{W}($$)$

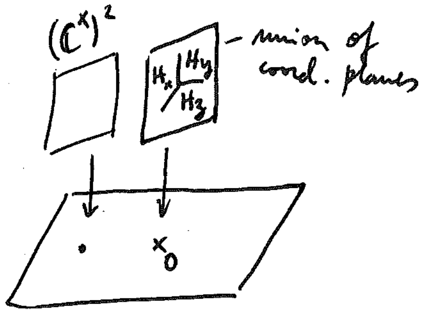
Actually, instead of $D^b\mathrm{Coh}(X)$ can take a higher dim'l (stabilized) LG model, with a more explicit 3-fold symmetry:

<u>stabilized mirror</u>: $X =$  $\longleftrightarrow \quad (X^\vee = \mathbb{C}^3,\ W = -xyz)$

$\mathcal{W}($$) \simeq D^b_{sing}(\mathbb{C}^3, -xyz)$

$= D^b\mathrm{Coh}(\{xyz=0\}) / \underset{0}{\underline{\mathrm{Perf}}}$

$$\begin{array}{c} \mathbb{C}^3 \\ {\scriptstyle -xyz}\downarrow \\ \mathbb{C} \end{array}$$



Study mod – $\mathbb{C}[x,y,z]/xyz$

$D^b\mathrm{Coh}(\{xyz=0\})/\mathrm{Perf}$ is gen'd by $\mathcal{O}_{H_x}, \mathcal{O}_{H_y}, \mathcal{O}_{H_z}$, w/ exact triangles coming from $\mathcal{O} \simeq 0$ (modding out by Perf), giving $\mathcal{O}_{H_x \cup H_y} \simeq \mathcal{O}_{H_z}[1]$

$$\begin{tikzcd} \mathcal{O}_{H_x} \arrow[rr] & & \mathcal{O}_{H_z} \arrow[dl] \\ & \mathcal{O}_{H_y} \arrow[ul] \arrow[u, phantom, "\text{exact}"] & \end{tikzcd}$$

$$0 \to \mathcal{O}_{H_x} \to \underset{\substack{\wr\wr \text{ in } D^b_{sing} \\ \mathcal{O}_{H_z}[1]}}{\mathcal{O}_{H_x \cup H_y}} \to \mathcal{O}_{H_y} \to 0$$

<u>Orlov</u>: $\leftrightarrow$ MF$(-xyz)$, $\mathbb{K}[x,y,z] \underset{yz}{\overset{-x}{\rightleftarrows}} \mathbb{K}[x,y,z]$

(MF are more delicate in a space that is not affine).

"LG stabilization"

Thm (Orlov): $D^b_{sing}(\mathbb{C}^3, -xyz) \simeq D^b Coh(\{xy=0\})$

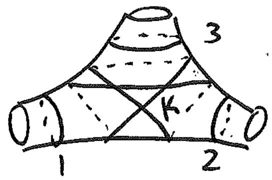
Knörrer periodicity: $(\underset{x,y}{Y} \times \underset{z}{\mathbb{C}}, W = zf + g) \leftrightarrow (f^{-1}(0) \subset Y, g|_{f^{-1}(0)})$, $g \in \mathcal{O}(Y)$

Symplectic analogue: if $f^{-1}(0)$ smooth, not in our case..., then (in affine setting)

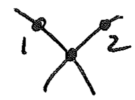
$FS(Y \times \mathbb{C}, W = zf) \leftrightarrow \mathcal{F}(f^{-1}(0) \subset Y)$

Morse-Bott, crit $W \simeq f^{-1}(0)$ $\quad$ (W?)

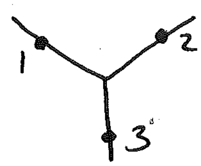
Compact Lags:



$\leftrightarrow$ skyscrapers of points



$\updownarrow$ stabil



(3 is not $\mathbb{Z}$-graded ...)

crit $W = \bigcup$ 3 coord axes

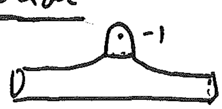
$[\mathcal{O}_{pt}]$ in D^b_{sing} $= A \oplus A[1]$

$K \underset{\text{Seidel}}{\longleftrightarrow} \mathcal{O}_{0 \in \mathbb{C}^3}$ in D^b_{sing}

$0 \to \mathcal{O}_A \oplus \mathcal{O}_B \to \mathcal{O} \to \mathcal{O}_0 \to 0$

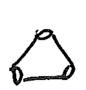
Compactification

(*) think of T as deformation parameter (Seidel, Sheridan) assoc. to divisor $\{-1\}$

$\mathbb{C}^* =$  $=$  $\cup \{-1\} \leftrightarrow (\mathbb{C}^3, \underbrace{-xyz + Tz}_{(T-xy)z})$

$(T-xy)z$, has MB sing along $\{xy = T\} \subset \mathbb{C}^2$, $\{xy=T\} \cong \mathbb{C}^*$, $\mathbb{K}^*$, if think of T as Novikov parameter (*)

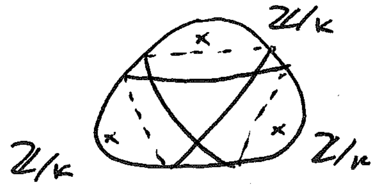
Similarly,

$\mathbb{CP}^1 =$  $\cup \{3 pts\} \leftrightarrow (\mathbb{C}^3, -xyz + Tx + Ty + Tz)$

Thm (Orlov): $D^b_{sing}(\mathbb{C}^3, -xyz + Tx + Ty + Tz) \simeq D^b_{sing}(\{xy = T\}, \underbrace{Tx + Ty}_{z}) \simeq$

$\simeq (\mathbb{C}^*, z + \frac{T^3}{z})$

Orbifold compactification:



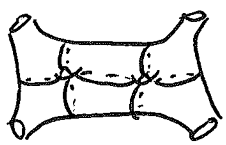
$\xleftrightarrow[\text{(Efimov)}]{\text{Seidel}}$ $(\mathbb{C}^3, -xyz + T(x^k + y^k + z^k))$

$k = 2g+1$ $\qquad \uparrow k:1$

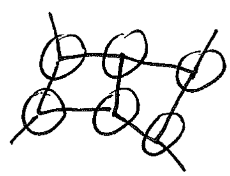
Σ_g  $\longleftrightarrow$ $(\ldots)/_{\mathbb{Z}/k}$

This gives mirror symmetry for Σ_g.

More generally: given a pair-of-pants decomposition



$\longleftrightarrow$ LG model w/ charts $(\mathbb{C}^3, -xyz)$



Čech model

$$W(-) \leftrightarrow D^b_{sing}$$

See Heather Lee's thesis.

<u>Next</u>: Higher dimensions.