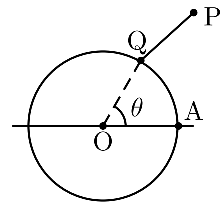
Math 53 – Practice Final

Problem 1. Given the points $P : (1, 1, -1)$, $Q : (1, 2, 0)$, $R : (-2, 2, 2)$, find

a) $\overrightarrow{PQ} \times \overrightarrow{PR}$; b) a plane $ax + by + cz = d$ through P, Q, R.

Problem 2. The roll of Scotch tape shown has outer radius a and is fixed in position (i.e., does not turn). Its end P is originally at the point A; the tape is then pulled from the roll so the free portion makes a 45-degree angle with the horizontal.



Write parametric equations $x = x(\theta)$, $y = y(\theta)$ for the curve C traced out by the point P as it moves. (Use vectors; θ is the angle shown).

Problem 3. The position vector of a point P is $\vec{r} = \langle 3\cos t, 5\sin t, 4\cos t\rangle$.

a) Show its speed is constant.

b) At what point(s) does P pass through the yz-plane?

Problem 4. Let $w = x^2y - xy^3$, and $P = (2, 1)$.

a) Find the directional derivative $D_{\hat{u}}w$ at P in the direction of $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$.

b) If you start at P and go a distance 0.01 in the direction of $\vec{A}$, by approximately how much will w change? (Give a decimal with one significant digit.)

Problem 5. a) Find the tangent plane at $(1, 1, 1)$ to the surface $x^2 + 2y^2 + 2z^2 = 5$; give the equation in the form $ax + by + cz = d$ and simplify the coefficients.

b) What dihedral angle does the tangent plane make with the xy-plane? (Hint: consider the normal vectors of the two planes.)

Problem 6. Find the minimum of the function $f(x, y) = x^2 + xy + y^2 - 4x - 5y + 7$ in the first quadrant. (Justify your answer.)

Problem 7. Find the point on the plane $2x + y - z = 6$ which is closest to the origin, by using Lagrange multipliers. (Minimize the square of the distance.)

Problem 8. a) Find the Lagrange multiplier equations for the point of the surface

$$x^4 + y^4 + z^4 + xy + yz + zx = 6$$

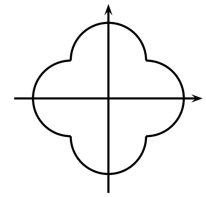
at which x is largest. (Do not solve.)

b) Given that x is largest at the point (x_0, y_0, z_0), find the equation for the tangent plane to the surface at that point.

Problem 9. Suppose that x, y, z are constrained by the equation $g(x, y, z) = 3$. Assume that at the point $P : (0, 0, 0)$ we have $g = 3$ and $\nabla g = \langle 2, -1, -1\rangle$. The equation $g = 3$ implicitly defines z as a function of x and y. Find the value of $\partial z/\partial x$ at P.

Problem 10. Evaluate by changing the order of integration: $\displaystyle\int_0^3 \int_{x^2}^9 xe^{-y^2}\, dy\, dx.$

Problem 11. A plane region R is bounded by four semicircles of radius 1, having ends at $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ and centers at $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$.



Set up an iterated integral in polar coordinates for the moment of inertia of R about the origin; take the density $\rho = 1$. Supply integrand and limits, but *do not evaluate* the integral. Use symmetry to simplify the limits of integration.

Problem 12. a) In the xy-plane, let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$. Express in terms of P and Q (and dx and dy) the line integral representing the flux of $\vec{F}$ across a simple closed curve C, with outward pointing normal vector.

b) Let $\vec{F} = ax\hat{\imath} + by\hat{\jmath}$. How should the constants a and b be related if the flux of $\vec{F}$ over any simple closed curve C is equal to the area inside C?

Problem 13. A solid hemisphere of radius 1 has its flat base on the xy-plane and center at the origin. Its density is equal to 1. Using an integral in spherical coordinates, find the z-coordinate of its center of mass.

Problem 14. Evaluate $\int_C (y-x)\,dx + (y-z)\,dz$ over the line segment C from $P : (1,1,1)$ to $Q : (2,4,8)$.

Problem 15. a) Let $\vec{F} = ay^2\hat{\imath} + 2y(x+z)\hat{\jmath} + (by^2+z^2)\hat{k}$. For what values of the constants a and b will $\vec{F}$ be conservative? Show work.

b) Using these values, find a function $f(x,y,z)$ such that $\vec{F} = \nabla f$.

c) Using these values, give the equation of a surface S having the property that $\displaystyle\int_P^Q \vec{F}\cdot d\vec{r} = 0$ for any two points P and Q on the surface S.

Problem 16. Let S be the closed surface whose bottom face B is the unit disc in the xy-plane and whose upper surface U is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the flux of $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ across U by using the divergence theorem.

Problem 17. Using the same data as in the preceding problem, calculate the flux of $\vec{F}$ across U directly, by setting up the surface integral for the flux and evaluating the resulting double integral.

Problem 18. Consider a surface S in 3-space given by an equation $z = f(x)$ (involving x and z alone, not y; its section by any plane $y = c$ is always the same curve.)

Show that if $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + xz\hat{k}$, then $\oint_C \vec{F}\cdot d\vec{r} = 0$ for any simple closed curve C lying on the surface S. (Use Stokes' theorem.)

Problem 19. Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying above the plane $z = 1$. Orient S upwards, and give its bounding circle C (lying in the plane $z = 1$) the compatible orientation.

a) Parametrize C and use this parametrization to evaluate the line integral $I = \oint_C xz\,dx + y\,dy + y\,dz$.

b) Compute the curl of the vector field $\vec{F} = xz\hat{\imath} + y\hat{\jmath} + y\hat{k}$.

c) Write down a flux integral through S which can be computed using the value of I.