

## Math 53 Practice Midterm 1 A – Solutions

### Problem 1.

$$\text{Area: } \int_0^{\pi/2} \frac{1}{2} (\sqrt{\sin 2\theta})^2 d\theta = \int_0^{\pi/2} \frac{1}{2} \sin(2\theta) d\theta = -\frac{1}{4} \cos(2\theta) \Big|_0^{\pi/2} = -\frac{1}{4}((-1) - 1) = \frac{1}{2}.$$

### Problem 2.

$$\text{a) } \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \hat{i} + \hat{j} + 2\hat{k}. \quad \text{Area} = \frac{1}{2} |\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| = \frac{1}{2} \sqrt{6}.$$

$$\text{b) Normal vector: } \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \hat{i} + \hat{j} + 2\hat{k}. \quad \text{Equation: } x + y + 2z = 3.$$

$$\text{c) Parametric equations for the line: } x = -1 + t, y = t, z = t.$$

$$\text{Substituting: } -1 + 4t = 3, t = 1, \text{ intersection point } (0, 1, 1).$$

### Problem 3.

$$\text{a) } \frac{d}{dt}(\vec{r} \cdot \vec{r}) = \vec{v} \cdot \vec{r} + \vec{r} \cdot \vec{v} = 2\vec{r} \cdot \vec{v}.$$

$$\text{b) Assume } |\vec{r}| \text{ is constant: then } \frac{d}{dt}(\vec{r} \cdot \vec{r}) = 2\vec{r} \cdot \vec{v} = 0, \text{ i.e. } \vec{r} \perp \vec{v}.$$

$$\text{c) } \vec{r} \cdot \vec{v} = 0, \text{ so } \frac{d}{dt}(\vec{r} \cdot \vec{v}) = \vec{v} \cdot \vec{v} + \vec{r} \cdot \vec{a} = 0. \text{ Therefore } \vec{r} \cdot \vec{a} = -|\vec{v}|^2.$$

### Problem 4.

$$\text{a) By measuring, } \Delta h = 100 \text{ for } \Delta s \simeq 500, \text{ so } D_{\hat{u}}h \simeq \frac{\Delta h}{\Delta s} \simeq 0.2.$$

$$\text{b) } Q \text{ is the northernmost point on the curve } h = 2200; \text{ the vertical distance between consecutive level curves is about } 1/3 \text{ of the given length unit, so } \frac{\partial h}{\partial y} \simeq \frac{\Delta h}{\Delta y} \simeq \frac{-100}{1000/3} \simeq -0.3.$$

### Problem 5.

$$\text{a) } \nabla f = (y - 4x^3)\hat{i} + x\hat{j}; \text{ at } P, \nabla f = \langle -3, 1 \rangle.$$

$$\text{b) } \Delta w \simeq -3\Delta x + \Delta y.$$

### Problem 6.

$f(x, y, z) = x^3y + z^2 = 3$ : the normal vector is  $\nabla f = \langle 3x^2y, x^3, 2z \rangle = \langle 3, -1, 4 \rangle$ . The tangent plane is  $3x - y + 4z = 4$ .

### Problem 7.

$$\frac{\partial w}{\partial x} = f_u u_x + f_v v_x = y f_u + \frac{1}{y} f_v. \quad \frac{\partial w}{\partial y} = f_u u_y + f_v v_y = x f_u - \frac{x}{y^2} f_v. \quad (\text{chain rule})$$

**Problem 8.**

a) The volume is  $xyz = xy(1-x^2-y^2) = xy-x^3y-xy^3$ . Critical points:  $f_x = y-3x^2y-y^3 = 0$ ,  $f_y = x-x^3-3xy^2 = 0$ .

b) Assuming  $x > 0$  and  $y > 0$ , the equations can be rewritten as  $1-3x^2-y^2 = 0$ ,  $1-x^2-3y^2 = 0$ . Solution:  $x^2 = y^2 = 1/4$ , i.e.  $(x, y) = (1/2, 1/2)$ .

At this point,  $f_{xx} = -6xy = -3/2$ ,  $f_{yy} = -6xy = -3/2$ ,  $f_{xy} = 1 - 3x^2 - 3y^2 = -1/2$ . So  $f_{xx}f_{yy} - f_{xy}^2 > 0$ , and  $f_{xx} < 0$ , it is a local maximum.

c) The maximum of  $f$  lies either at  $(1/2, 1/2)$ , or on the boundary of the domain or at infinity. Since  $f(x, y) = xy(1-x^2-y^2)$ ,  $f = 0$  when either  $x \rightarrow 0$  or  $y \rightarrow 0$ , and  $f \rightarrow -\infty$  when  $x \rightarrow \infty$  or  $y \rightarrow \infty$  (since  $x^2 + y^2 \rightarrow \infty$ ). So the maximum is at  $(x, y) = (\frac{1}{2}, \frac{1}{2})$ , where  $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$ .