Math 53 Homework 7

Due Wednesday 3/9/16 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 2/29 – Midterm 1 Review

Wednesday 3/2 – MIDTERM 1

Chapter 14 – Complements

These two problems complete the material from Chapter 14, and can be attempted already, but are not directly related to the material on Midterm 1.

- **Work:** Problems 1 and 2 below.

Friday 3/4 – Lagrange multipliers

- **Read:** section 14.8.
- **Work:** 14.8: (1), 3[1], (7), 9[2], (29), 33, (37), 39.[3]
  Problem 3 below.
- **Bonus problem** (extra credit, hard): Problem 4 below.

Problem 1. – Least-squares interpolation.

In experimental sciences, statistics, and many other fields, one often wishes to establish a linear relationship between two quantities (say x and y). Repeated experiments (or sampling of x and y for various individuals among the general population) give a set of experimental data (x_1, y_1), ..., (x_n, y_n) (each pertaining to a different experiment or to a different individual). One then attempts to find the straight line $y = mx + b$ that best fits the given experimental data. Least-squares interpolation is the most common method for doing so. (Note: the goal is to find m and b, which describe the relation between x and y – we are *not* trying to solve for x and y!). We will first work out the general formula (following a problem in the book), then apply it in an example.

a) Do section 14.7 exercise # 59 (6th/7th ed: 14.7 # 55). You don't need to prove that the critical point is a minimum.

Hint: In this problem, the total square deviation $\sum_{i=1}^{n} d_i^2$ is a function of the two variables m and b; all the x_i and y_i are constants (as part of the given experimental data). So this is a min-max problem in two variables, just like those we have seen in class. Looking for a critical point should give you the two equations. Proving that

[1] **6th/7th ed:** do the 8th ed version: $f(x, y) = x^2 - y^2$, $x^2 + y^2 = 1$.
[2] **6th/7th ed:** do the 8th ed version: $f(x, y, z) = xy^2z$, $x^2 + y^2 + z^2 = 4$.
[3] **6th ed:** 14.8: (1), 3*, (7), 9*, (25), 29, (33), 35; **7th ed:** (1), 3*, (7), 9*, (27), 31, (35), 37.

the critical point is indeed a minimum can be done (using the second derivative test or other methods) but takes quite a bit of effort; doing it is strictly optional.

b) The growth of the number of active Uber drivers in the US is illustrated by the table below (source: Uber company data)[4]:

Month	June 2012	Dec 2012	June 2013	Dec 2013	June 2014	Dec 2014
Drivers (thousands)	1.3	6.3	11.7	34.2	71.4	162.0

To make the calculations easier, we take x to be the amount of time in years elapsed since 1/1/2010, so the given data correspond to x_i values of 2.5, 3, 3.5, 4, 4.5, 5. We take y_i to be the number of drivers in thousands. So $(x_1, y_1) = (2.5, 1.3), \ldots,$ $(x_6, y_6) = (5, 162)$.

Use the result of part (a) to write down the equations satisfied by the slope m and the intercept b of the best-fit line for this data. Next, solve these equations and give the values of m and b. (Use a calculator or a computer!)

Compare the predicted values $y = mx + b$ with the actual data for June 2012 and December 2014. How good is the linear fit?

c) When a new technology becomes available, at first its use tends to grow exponentially rather than linearly. Therefore, we will try to find a best line fit for $Y = \ln y$ as a function of x, in the form $Y = Mx + B$. So the experimental data are now (x_i, Y_i) where $Y_i = \ln y_i$: $(x_1, Y_1) = (2.5, 0.262)$, etc.

Write down the equations satisfied by M and B for this data, and solve them numerically. (Note: this is similar to part (b) except we use Y_i instead of y_i.)

If you exponentiate the equation $\ln(y) = Mx + B$, you get $y = e^B e^{Mx}$. Compute the predicted values of y for June 2012 and December 2014, and compare them with the actual data. Is the exponential fit better than the linear fit?

d) Optional: produce a plot that shows the given data points and the linear and exponential best fit curves.

e) According to the exponential best fit, how many Uber drivers were there in September 2015 ($x = 5.75$)?[5] How many will there be in December 2019 ($x = 10$)? (For comparison the US population is projected to be about 340 million by then; this illustrates that exponential growth is not sustainable over long periods of time.)

Problem 2. – Non-independent variables.

The goal of this problem is to illustrate a subtlety in the definition of partial derivatives when variables are not independent. This is an important issue in thermodynamics and some other fields; here we consider just a simple mathematical example.

[4] https://s3.amazonaws.com/uber-static/comms/PDF/Uber_Driver-Partners_Hall_Kreuger_2015.pdf Uber defines a driver as "active" if they have provided at least 4 trips in the given month.

[5] There is no publicly available 2015 data, but various media stories from mid-October 2015 state that Uber had 327,000 active drivers at that point, while on 11/4/2015 an Uber board member said in an interview that the number was over 400,000.

Let $w = x^2 + y^2 + z^2$, where the variables x, y, z are related to each other by the equation $y^2 + xz = 2$. We can give three different meanings to the quantity $\partial w/\partial x$.

(i) We can treat the variables x, y, z as independent, and write $w = f(x, y, z) = x^2 + y^2 + z^2$. Then we consider $\partial f/\partial x$.

(ii) We can treat x and y as independent variables, with z implicitly defined as a function of x and y by the relation $y^2 + xz = 2$. Then w is given by some function $g(x, y)$, and we consider $\dfrac{\partial g}{\partial x}$. This quantity is sometimes denoted by $\left(\dfrac{\partial w}{\partial x}\right)_y$.

(iii) We can treat x and z as independent variables, with y implicitly defined as a function of x and z by the relation $y^2 + xz = 2$. Then w is given by some function $h(x, z)$, and we consider $\dfrac{\partial h}{\partial x}$. This quantity is sometimes denoted by $\left(\dfrac{\partial w}{\partial x}\right)_z$.

a) Determine the functions f, g, h, and calculate $\partial f/\partial x$, $\partial g/\partial x$, and $\partial h/\partial x$. Also say in each case which quantities are being held constant and which ones are not.

(Optional: compare the values of these partial derivatives at $(x, y, z) = (1, 1, 1)$ to convince yourself that they are really different.)

b) Now we try a more systematic approach, which would work even if we were unable to find expressions for the functions $g(x, y)$ and $h(x, z)$ by solving the constraint equation.

First, express the differential dw in terms of dx, dy and dz, and also differentiate the constraint equation to find a relation between dx, dy and dz. Then, use this relation to eliminate dz and express dw in terms of dx and dy; use this to find $\left(\dfrac{\partial w}{\partial x}\right)_y$. Similarly, eliminate dy to find $\left(\dfrac{\partial w}{\partial x}\right)_z$.

(Your answers might be different from those in part (a), but are they consistent with them?)

Problem 3. This problem uses Lagrangian multipliers to find an exact answer to Problem 2(b) of HW 6.

a) Use the method of Lagrange multipliers to write down the system of equations satisfied by the point closest to $(4, 2, 3)$ at which $x^2 + y^2 - 6z = 0$. (Hint: it is easier to minimize the square of the distance).

b) Solve the equations you found in (a) to get the exact location of the point. Then use a calculator to evaluate your answers to five decimal places. Compare your answers with the approximate solution you found in HW 6 Problem 2(b). Was each coordinate of the approximate answer within 1/100 of the exact answer?

Problem 4 (extra credit, hard)

UC Berkeley has hired a famous architect to design a new building, to be located on a flat, triangular plot of land, with sides of given lengths a_1, a_2, a_3. The building will have the shape of a pyramid, with the base exactly covering the entire plot. The volume of the pyramid is also fixed (in order for the building to accommodate the planned amount of occupants). The architect has decided that the three triangular side faces of the pyramid would be entirely covered in gold. However, to minimize cost, the shape of the pyramid will be chosen so that the sum of the areas of the side faces is the smallest possible one (given the fixed triangular base and fixed volume). The goal of this problem is to find the optimal position of the apex P of the pyramid (the point at the tip).

a) One possibility would be to work in coordinates, expressing the surface area in terms of the coordinates (x, y, z) of the point P, and those of the vertices of the base triangle (which lies in the xy-plane). What is the constraint satisfied by x, y, z? Unfortunately, solving the problem in this way leads to extremely lengthy calculations. (Convince yourself of this! How would you express the area in terms of x, y, z? Don't write the entire formula).

b) The problem is easier to solve if one uses a different set of variables. Denote by Q the point in the base triangle that lies directly beneath P, and let u_1, u_2, u_3 be the distances from Q to the sides of lengths a_1, a_2, a_3 of the triangle respectively. What is the relation between u_1, u_2, u_3? (Hint: decompose the base triangle into 3 smaller triangles with a vertex at Q). What is the total area of the side faces of the pyramid as a function of u_1, u_2, u_3?

c) Use Lagrange multipliers to solve the min/max problem you arrived at in part (b). What can you say about the values of u_1, u_2, u_3 at the solution? Geometrically, what does this say about the point Q?