Math 53 Practice Midterm 2 A – Solutions

1. The area of the triangle is 2, so $\bar{y} = \frac{1}{2}\int_0^1 \int_{2y-2}^{2-2y} y\,dxdy$.

2. $\rho = |x| = r|\cos\theta|$. Using symmetry, $I_0 = \iint_D r^2\,\rho\,r dr d\theta =$

$$\int_0^{2\pi}\int_0^1 r^2 |r\cos\theta| r dr d\theta = 4\int_0^{\pi/2}\int_0^1 r^4\cos\theta dr d\theta = 4\int_0^{\pi/2}\frac{1}{5}\cos\theta d\theta = \frac{4}{5}.$$

3. $x = x$, $y = x^2$, $0 \le x \le 1$: so $\int_C yx^3dx + y^2dy = \int_0^1 x^2x^3dx + (x^2)^2(2xdx) = \int_0^1 3x^5dx = \frac{1}{2}$.

4. a) $Q_x = 6x^2 + by^2$, $P_y = ax^2 + 3y^2$. $Q_x = P_y$ provided $a = 6$ and $b = 3$.

b) $f_x = 6x^2y + y^3 + 1 \Rightarrow f = 2x^3y + xy^3 + x + g(y)$. Therefore, $f_y = 2x^3 + 3xy^2 + g'(y)$. Comparing this with Q, we get $2x^3 + 3xy^2 + g'(y) = 2x^3 + 3xy^2 + 2$ so $g'(y) = 2$ and $g = 2y + c$. So

$$f = 2x^3y + xy^3 + x + 2y \quad (+\text{constant}).$$

c) C starts at $(1,0)$ and ends at $(-e^\pi, 0)$, so $\int_C \vec{F}\cdot d\vec{r} = f(-e^\pi, 0) - f(1,0) = -e^\pi - 1$.

5. a) $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = 3x^2/y$. Therefore, $dudv = |3x^2/y|\,dxdy = 3|u|\ dxdy$ and hence $dx\,dy = \frac{1}{3|u|}\,du\,dv$.

b) $\iint_R dx\,dy = \int_2^4\int_1^5 \frac{1}{3u}dudv = \int_2^4 \frac{1}{3}\ln 5\ dv = \frac{2}{3}\ln 5$.

6. a) $\oint_C M dx = \iint_R -M_y\,dA$. (Green's theorem)

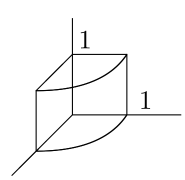
b) We want M such that $-M_y = (x+y)^2$. We can use e.g. $M = -\frac{1}{3}(x+y)^3$.

7. a) $\operatorname{div}\vec{F} = 2y$, so $\oint_C \vec{F}\cdot\hat{\mathbf{n}}\,ds = \iint_R 2y\,dA = \int_0^1\int_0^{x^3} 2y\,dydx = \int_0^1 x^6dx = \frac{1}{7}$.

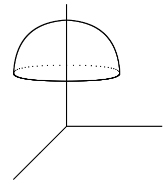
b) For the flux through C_1, $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$ implies $\vec{F}\cdot\hat{\mathbf{n}} = -(1+y^2) = -1$ where $y = 0$. The length of C_1 is 1, so the total flux through C_1 is $\int_{C_1}(-1)\,ds = -1$. The flux through C_2 is zero because $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ and $\vec{F}\perp\hat{\mathbf{i}}$.

c) $\int_{C_3}\vec{F}\cdot\hat{\mathbf{n}}\,ds = \int_{C_1+C_2+C_3}\vec{F}\cdot\hat{\mathbf{n}}\,ds - \int_{C_1}\vec{F}\cdot\hat{\mathbf{n}}\,ds - \int_{C_2}\vec{F}\cdot\hat{\mathbf{n}}\,ds = \frac{1}{7} - (-1) - 0 = \frac{8}{7}$.

8. $\int_0^{\pi/2}\int_0^1\int_0^1 r^2\,r\,dz\,dr\,d\theta$.



9. Sphere: $\rho = 2a\,\cos\phi$; plane: $\rho = a\,\sec\phi$.



Hence: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{a\,\sec\phi}^{2a\,\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

10. a) S is the graph of $z = f(x,y) = 1 - x^2 - y^2$, so $\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy = \langle 2x, 2y, 1\rangle\,dx\,dy$.

Therefore $\iint_S \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_S \langle x, y, 2(1-z)\rangle \cdot \langle 2x, 2y, 1\rangle\,dx\,dy = \iint_S 2x^2 + 2y^2 + 2(1-z)\,dx\,dy = \iint_S 4x^2 + 4y^2\,dx\,dy$ (since $z = 1 - x^2 - y^2$).

Shadow = unit disc $x^2 + y^2 \le 1$; switching to polar coordinates, we have

$\iint_S \vec{F}\cdot\hat{\mathbf{n}}\,dS = \int_0^{2\pi}\int_0^1 4r^2\,r\,dr\,d\theta = \int_0^{2\pi}\left[r^4\right]_0^1 d\theta = 2\pi.$

b) Let $T =$ unit disc in the xy-plane, with normal vector pointing down ($\hat{\mathbf{n}} = -\hat{\mathbf{k}}$). Then

$\iint_T \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_T \langle x, y, 2\rangle \cdot (-\hat{\mathbf{k}})\,dS = \iint_T -2\,dS = -2\,\mathrm{Area} = -2\pi$. By divergence theorem,

$\iint_{S+T} \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV = 0$, since $\operatorname{div}\vec{F} = 1 + 1 - 2 = 0$. Therefore $\iint_S = -\iint_T = +2\pi$.