Math 53 – Practice Midterm 2 A – 80 minutes

Problem 1. (8 points) Let $(\bar{x}, \bar{y})$ be the center of mass of the triangle with vertices at $(-2, 0)$, $(0, 1)$, $(2, 0)$ and uniform density $\rho = 1$.

Write an integral formula for $\bar{y}$. Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.

Problem 2. (8 points) Find the polar moment of inertia $I_0 = \iint (x^2 + y^2)\, \rho \, dA$ of the unit disk with density ρ equal to the distance from the y-axis.

Problem 3. (7 points) For $\vec{F} = yx^3\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$, find $\int_C \vec{F} \cdot d\vec{r}$ on the portion of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Problem 4. (10 points) Consider the vector field $\vec{F} = (ax^2y + y^3 + 1)\hat{\mathbf{i}} + (2x^3 + bxy^2 + 2)\hat{\mathbf{j}}$, where a and b are constants.

a) (3) Find the values of a and b for which $\vec{F}$ is conservative.

b) (4) For these values of a and b, find $f(x, y)$ such that $\vec{F} = \nabla f$. (Use a systematic method and show your work.)

c) (3) Still using the values of a and b from part (a), compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ along the curve C given by the parametric equations $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi$.

Problem 5. (10 points) Consider the region R in the first quadrant bounded by the curves $y = x^2$, $y = x^2/5$, $xy = 2$, and $xy = 4$.

a) (5) Compute $dxdy$ in terms of $dudv$ if $u = x^2/y$ and $v = xy$.

b) (5) Express the area of R as a double integral in uv coordinates and evaluate it.

Problem 6. (7 points)

a) (3) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x, y)$. Express $\displaystyle\oint_C M\, dx$ as a double integral over R.

b) (4) Find M so that $\displaystyle\oint_C M dx$ is the mass of R with density $\rho(x, y) = (x + y)^2$.

Problem 7. (15 points) Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$.

a) (5) Use Green's theorem to find the flux $\oint \vec{F} \cdot \hat{\mathbf{n}}\, ds$ of $\vec{F} = (1 + y^2)\hat{\mathbf{j}}$ out of R.

b) (7) Find the flux of $\vec{F}$ out of R through the two sides C_1 (the horizontal segment) and C_2 (the vertical segment).

c) (3) Use parts (a) and (b) to find the flux out of the third side C_3.

Problem 8. (8 points) Let C be the portion of the cylinder $x^2+y^2 \leq 1$ lying in the first octant $(x \geq 0,\ y \geq 0,\ z \geq 0)$ and below the plane $z = 1$. Set up a triple integral in *cylindrical coordinates* which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$. (Recall $I_z = \iiint (x^2 + y^2)\,\delta\,dV$.)

(Give integrand and limits of integration, but *do not evaluate*.)

Problem 9. (10 points)

A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Set up a triple integral in *spherical coordinates* which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but *do not evaluate*.)

Problem 10. (17 points) Let S be the surface formed by the portion of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 2(1-z)\hat{\mathbf{k}}$.

Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) (10) by direct calculation of $\iint_S \vec{F} \cdot \hat{\mathbf{n}}\,dS$;

b) (7) by computing the flux of $\vec{F}$ across a simpler surface and using the divergence theorem.