# Math 53 Midterm 2 Wednesday, November 15, 2017 2:10 - 3:00

**Directions:** Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. **No books, notes, calculators, or electronic devices**. Please stop when asked to and don't talk until your paper is handed in.

**YOUR NAME:** SOLUTIONS

**Last 4 digits of your student ID:** ______

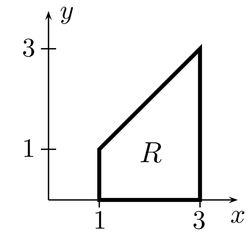
**Please mark the box next to your GSI's name, and circle your discussion section:**

| | | | |
|---|---|---|---|
| **Discussion section:** | ☐ Nima MOINI | # 101 (8am), | # 103 (9am) |
| | ☐ Ritwik GHOSH | # 102 (8am), | # 104 (9am) |
| | ☐ Kyeonsik NAM | # 105 (10am), | # 108 (11am) |
| | ☐ Jasper DENG | # 106 (10am), | # 107 (11am) |
| | ☐ Michael YEH | # 109 (12pm), | # 111 (1pm) |
| | ☐ Jingyi WANG | # 110 (12pm), | # 112 (1pm) |
| | ☐ Jeffrey HICKS | # 113 (3pm), | # 115 (4pm) |
| | ☐ Justin BRERETON | # 114 (3pm), | # 116 (5pm) |
| | ☐ Ahmad ZAREEI | # 118 (4pm), | # 117 (5pm) |

| GRADING | |
|---|---|
| 1. ______ | /25 |
| 2. ______ | /20 |
| 3. ______ | /25 |
| 4. ______ | /30 |
| TOTAL | /100 |

**Problem 1.** (25) Let $R$ be the quadrilateral with vertices $(1,0)$, $(3,0)$, $(3,3)$ and $(1,1)$ (see picture). Set up the double integral $\iint_R x^2\,dA$ as an iterated integral – give the integrand and bounds of integration, but <u>*do not evaluate*</u>.

a) (5) in rectangular $(x,y)$ coordinates:




$$\int_1^3 \int_0^x x^2\,dy\,dx$$

$$\left(\text{or } \int_0^1 \int_1^3 x^2\,dx\,dy + \int_1^3 \int_y^3 x^2\,dx\,dy\right)$$

b) (10) in polar coordinates:

$$x = 1 \iff r\cos\theta = 1 \iff r = \sec\theta$$
$$x = 3 \iff r\cos\theta = 3 \iff r = 3\sec\theta$$
$$y = x \iff \theta = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{3\sec\theta} (r\cos\theta)^2\, r\,dr\,d\theta$$

c) (10) using the variables $u = x$ and $v = y/x$:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x}, \text{ so} \quad du\,dv = \frac{1}{x}\,dx\,dy$$
$$dx\,dy = x\,du\,dv = u\,du\,dv$$

$$R:\ 1 \le x \le 3 \iff 1 \le u \le 3$$
$$0 \le y \le x \iff 0 \le v \le 1$$

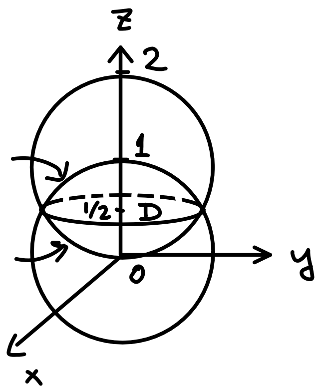
So $$\iint_R x^2\,dA = \int_0^1 \int_1^3 u^3\,du\,dv \qquad \left(\text{or } \int_1^3 \int_0^1 u^3\,dv\,du\right).$$

**Problem 2.** (20) Let $D$ be the intersection of the solid spheres of radius 1 centered at the origin and at (0,0,1), namely the solid consisting of all points where $x^2+y^2+z^2 \le 1$ and $x^2+y^2+(z-1)^2 \le 1$. Set up the triple integral $\iiint_D z^2\,dV$ using iterated integrals; give the integrand and bounds but <u>do not evaluate</u>.

a) (10) in cylindrical coordinates:

top: $r^2+z^2=1$
$z=\sqrt{1-r^2}$

bottom: $r^2+(z-1)^2=1$
$z=1-\sqrt{1-r^2}$




Intersection: $z=1/2$, $r=\frac{\sqrt{3}}{2}$.

$$\iiint_D z^2\,dV = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} z^2\, r\,dz\,dr\,d\theta$$

or

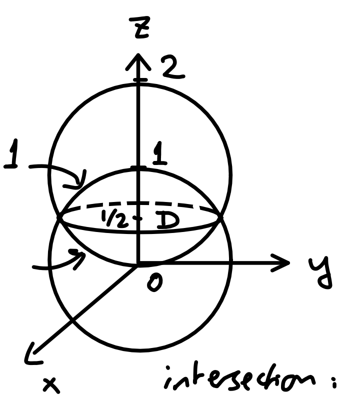
$$\int_0^{\frac{1}{2}}\int_0^{2\pi}\int_0^{\sqrt{1-(z-1)^2}} z^2\, r\,dr\,d\theta\,dz + \int_{\frac{1}{2}}^{1}\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} z^2\, r\,dr\,d\theta\,dz$$

b) (10) in spherical coordinates:

$\rho=1$

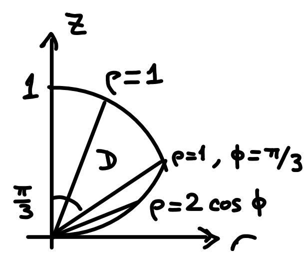
$\rho=2\cos\phi$

(seen in class; from geometry or from equation $x^2+y^2+z^2-2z=0$)
$\rho^2-2\rho\cos\phi=0$




intersection:
$2\cos\phi=1$
$\phi=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$.

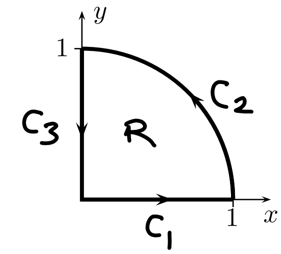
cross.section:




$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1}(\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi/2}\int_0^{2\cos\phi}(\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

**Problem 3.** (25) Let $\vec{F} = 3(x^2 + y^2)\hat{\jmath}$ and let $C$ be the closed curve consisting of the $x$ axis from the origin to $(1, 0)$, the portion of the unit circle in the first quadrant from $(1, 0)$ to $(0, 1)$, and the $y$ axis back from $(0, 1)$ to the origin (see picture). Calculate $\int_C \vec{F} \cdot d\vec{r}$:

a) (15) directly as a line integral,




$C_1$: $0 \le x \le 1$
$y = 0$, $dy = 0$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} 3(x^2+y^2)\,dy = 0.$$

$C_2$: $x = \cos\theta$, $dx = -\sin\theta\,d\theta$ $\qquad 0 \le \theta \le \frac{\pi}{2}$
$y = \sin\theta$, $dy = \cos\theta\,d\theta$
$x^2+y^2 = 1$

$$\int_{C_2} 3(x^2+y^2)\,dy = \int_{C_2} 3\,dy = \int_0^{\frac{\pi}{2}} 3\cos\theta\,d\theta = \Big[3\sin\theta\Big]_0^{\pi/2} = 3.$$

$C_3$: $x = 0$, $y$ from 1 to 0

$$\int_{C_3} 3(x^2+y^2)\,dy = \int_1^0 3y^2\,dy = -\int_0^1 3y^2\,dy = -\Big[y^3\Big]_0^1 = -1.$$

Total: $\oint_C \vec{F} \cdot d\vec{r} = 0 + 3 - 1 = 2.$

b) (10) using Green's theorem.

$$\oint_C 3(x^2+y^2)\,dy = \iint_R \frac{\partial}{\partial x}\big(3(x^2+y^2)\big)\,dA$$
$$= \iint_R 6x\,dA$$
$$= \int_0^{\pi/2}\int_0^1 6r\cos\theta\; r\,dr\,d\theta$$

Inner: $\Big[2r^3\cos\theta\Big]_0^1 = 2\cos\theta$

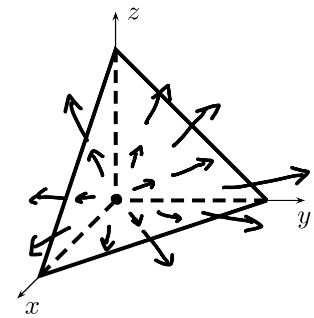
Outer: $\int_0^{\pi/2} 2\cos\theta\,d\theta = \Big[2\sin\theta\Big]_0^{\pi/2} = 2.$

**Problem 4.** (30) Consider the solid tetrahedron bounded by the plane $x + y + z = 1$ in the first octant, with vertices at the origin, (1,0,0), (0,1,0) and (0,0,1), and the vector field $\vec{F} = \langle x, y, z\rangle$.

a) (8) Without calculation, determine whether the flux of $\vec{F}$ out of the tetrahedron through each of the four faces (the triangle in the plane $x + y + z = 1$, and the three triangles in the coordinate planes) is positive, negative, or zero. [Justify your answer]

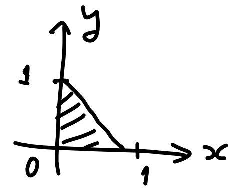
Back faces (xy, xz, yz planes): zero
($\vec{F}$ points radially outwards, tangent to S)

top face ($x+y+z=1$): positive ($\vec{F}$ points out of tetrahedron).




b) (12) Calculate the flux of $\vec{F}$ through the top face of the tetrahedron (the triangle with vertices (1,0,0), (0,1,0) and (0,0,1)) directly as a surface integral.

$z = 1 - x - y$ over triangular region



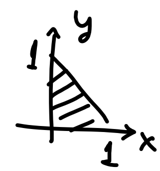

$$\hat{n}\, dS = \langle -z_x, -z_y, 1\rangle\, dx\, dy$$
$$= \langle 1, 1, 1\rangle\, dx\, dy$$

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \langle x, y, z\rangle \cdot \langle 1,1,1\rangle\, dx\, dy$$
$$= \iint_S (x+y+z)\, dx\, dy$$
$$= \int_0^1 \int_0^{1-x} 1\, dy\, dx.$$
(using $z = 1-x-y$)

Inner: $1-x$.
Outer: $\int_0^1 (1-x)\, dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$.

c) (10) Calculate the total flux of $\vec{F}$ out of the tetrahedron by using the divergence theorem.

$0 \le z \le 1-x-y$ above




$\operatorname{div} \vec{F} = 1 + 1 + 1 = 3$, so

$$\oiint_S \vec{F}\cdot\hat{n}\, dS = \iiint_R 3\, dV = 3\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$
$$= 3\int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$

Inner: $3\left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} = \frac{3}{2}(1-x)^2$

Outer: $\int_0^1 \frac{3}{2}(1-x)^2\, dx = \left[-\frac{1}{2}(1-x)^3\right]_0^1 = \frac{1}{2}$.

(or: 3 vol (tetrahedron)
$= 3\cdot\left(\frac{1}{3}\cdot \text{base}\cdot\text{height}\right)$
$= 3\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{2}$).