

**Directions:** Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. **No books, notes, calculators, or electronic devices.** Please stop when asked to and don't talk until your paper is handed in.

YOUR NAME:           SOLUTIONS          

Last 4 digits of your student ID: \_\_\_\_\_

Please mark the box next to your GSI's name, and circle your discussion section.

<b>Discussion section:</b>	<input type="checkbox"/>	Nima MOINI	# 101 (8am),	# 103 (9am)
	<input type="checkbox"/>	Ritwik GHOSH	# 102 (8am),	# 104 (9am)
	<input type="checkbox"/>	Kyeonsik NAM	# 105 (10am),	# 108 (11am)
	<input type="checkbox"/>	Jasper DENG	# 106 (10am),	# 107 (11am)
	<input type="checkbox"/>	Michael YEH	# 109 (12pm),	# 111 (1pm)
	<input type="checkbox"/>	Jingyi WANG	# 110 (12pm),	# 112 (1pm)
	<input type="checkbox"/>	Jeffrey HICKS	# 113 (3pm),	# 115 (4pm)
	<input type="checkbox"/>	Justin BRERETON	# 114 (3pm),	# 116 (5pm)
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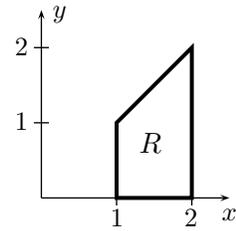
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**Problem 1.** (25) Let  $R$  be the quadrilateral with vertices  $(1,0)$ ,  $(2,0)$ ,  $(2,2)$  and  $(1,1)$  (see picture). Set up the double integral  $\iint_R x \, dA$  as an iterated integral – give the integrand and bounds of integration, but do not evaluate.

a) (5) in rectangular  $(x, y)$  coordinates:

$$\int_1^2 \int_0^x x \, dy \, dx$$

$$\text{(or } \int_0^1 \int_1^2 x \, dx \, dy + \int_1^2 \int_y^2 x \, dx \, dy \text{)}$$



b) (10) in polar coordinates:

$$x=1 \Leftrightarrow r \cos \theta = 1 \Leftrightarrow r = \sec \theta$$

$$x=2 \Leftrightarrow r \cos \theta = 2 \Leftrightarrow r = 2 \sec \theta$$

$$y=x \Leftrightarrow \theta = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} \int_{\sec \theta}^{2 \sec \theta} r \cos \theta \, r \, dr \, d\theta$$

c) (10) using the variables  $u = x$  and  $v = y/x$ :

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x}, \text{ so } \begin{aligned} du \, dv &= \frac{1}{x} \, dx \, dy \\ dx \, dy &= x \, du \, dv = u \, du \, dv \end{aligned}$$

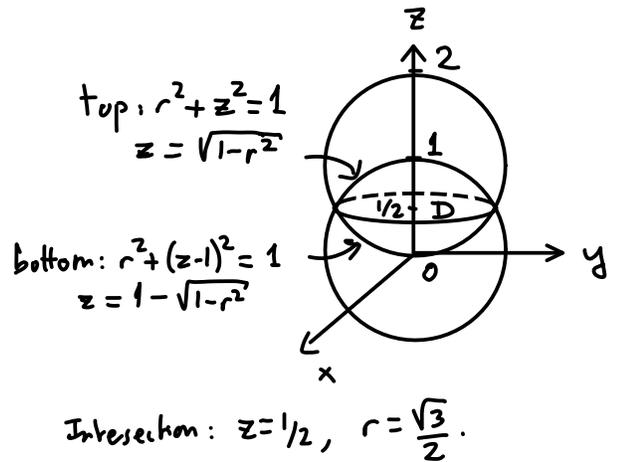
$$R: 1 \leq x \leq 2 \Leftrightarrow 1 \leq u \leq 2$$

$$0 \leq y \leq x \Leftrightarrow 0 \leq v \leq 1$$

$$\text{So } \iint_R x \, dA = \int_0^1 \int_1^2 u^2 \, du \, dv \quad \text{(or } \int_1^2 \int_0^1 u^2 \, dv \, du \text{)}.$$

**Problem 2.** (20) Let  $D$  be the intersection of the solid spheres of radius 1 centered at the origin and at  $(0,0,1)$ , namely the solid consisting of all points where  $x^2 + y^2 + z^2 \leq 1$  and  $x^2 + y^2 + (z-1)^2 \leq 1$ . Set up the triple integral  $\iiint_D z^4 dV$  using iterated integrals; give the integrand and bounds but do not evaluate.

a) (10) in cylindrical coordinates:

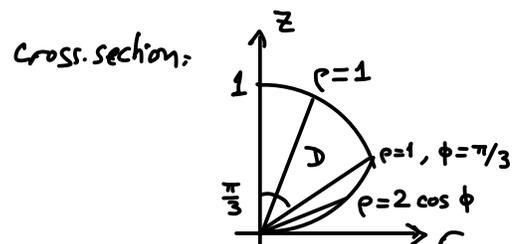
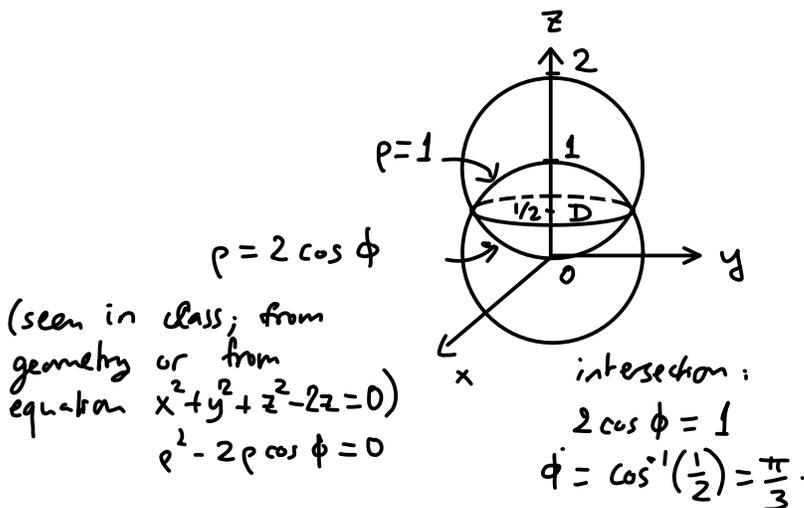


$$\iiint_D z^4 dV = \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} z^4 r dz dr d\theta$$

or

$$\int_0^{\frac{1}{2}} \int_0^{2\pi} \int_0^{\sqrt{1-(z-1)^2}} z^4 r dr d\theta dz + \int_{\frac{1}{2}}^1 \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} z^4 r dr d\theta dz$$

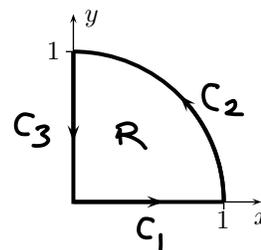
b) (10) in spherical coordinates:



$$\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 (\rho \cos \phi)^4 \rho^2 \sin \phi d\rho d\phi d\theta + \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\pi/2} \int_0^{2 \cos \phi} (\rho \cos \phi)^4 \rho^2 \sin \phi d\rho d\phi d\theta$$

**Problem 3.** (25) Let  $\vec{F} = (x^2 + y^2)\hat{j}$  and let  $C$  be the closed curve consisting of the  $x$  axis from the origin to  $(1, 0)$ , the portion of the unit circle in the first quadrant from  $(1, 0)$  to  $(0, 1)$ , and the  $y$  axis back from  $(0, 1)$  to the origin (see picture). Calculate  $\int_C \vec{F} \cdot d\vec{r}$ :

a) (15) directly as a line integral,



$$C_1: \begin{array}{l} 0 \leq x \leq 1 \\ y=0, \quad dy=0 \end{array} \quad \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} (x^2 + y^2) dy = 0.$$

$$C_2: \begin{array}{l} x = \cos \theta, \quad dx = -\sin \theta d\theta \\ y = \sin \theta, \quad dy = \cos \theta d\theta \\ x^2 + y^2 = 1 \end{array} \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\int_{C_2} (x^2 + y^2) dy = \int_{C_2} 1 \cdot dy = \int_0^{\pi/2} \cos \theta d\theta = [\sin \theta]_0^{\pi/2} = 1.$$

$$C_3: x=0, \quad y \text{ from } 1 \text{ to } 0$$

$$\int_{C_3} (x^2 + y^2) dy = \int_1^0 y^2 dy = -\int_0^1 y^2 dy = -\left[\frac{y^3}{3}\right]_0^1 = -\frac{1}{3}.$$

$$\text{Total: } \oint_C \vec{F} \cdot d\vec{r} = 0 + 1 - \frac{1}{3} = \frac{2}{3}.$$

b) (10) using Green's theorem.

$$\begin{aligned} \oint_C (x^2 + y^2) dy &= \iint_R \frac{\partial}{\partial x} (x^2 + y^2) dA \\ &= \iint_R 2x dA \\ &= \int_0^{\pi/2} \int_0^1 2r \cos \theta r dr d\theta \end{aligned}$$

$$\text{Inner: } \left[ \frac{2}{3} r^3 \cos \theta \right]_0^1 = \frac{2}{3} \cos \theta$$

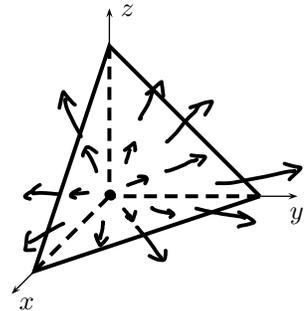
$$\text{Outer: } \int_0^{\pi/2} \frac{2}{3} \cos \theta d\theta = \left[ \frac{2}{3} \sin \theta \right]_0^{\pi/2} = \frac{2}{3}.$$

**Problem 4.** (30) Consider the solid tetrahedron bounded by the plane  $x + y + z = 1$  in the first octant, with vertices at the origin,  $(1,0,0)$ ,  $(0,1,0)$  and  $(0,0,1)$ , and the vector field  $\vec{F} = \langle x, y, z \rangle$ .

a) (8) Without calculation, determine whether the flux of  $\vec{F}$  out of the tetrahedron through each of the four faces (the triangle in the plane  $x + y + z = 1$ , and the three triangles in the coordinate planes) is positive, negative, or zero. [Justify your answer]

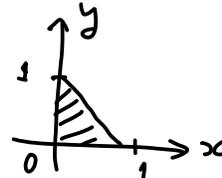
Back faces ( $xy, xz, yz$  planes): zero  
 ( $\vec{F}$  points radially outwards, tangent to  $S$ )

top face ( $x+y+z=1$ ): positive ( $\vec{F}$  points out of tetrahedron).



b) (12) Calculate the flux of  $\vec{F}$  through the top face of the tetrahedron (the triangle with vertices  $(1,0,0)$ ,  $(0,1,0)$  and  $(0,0,1)$ ) directly as a surface integral.

$z = 1 - x - y$  over triangular region



$$\begin{aligned} \hat{n} dS &= \langle -z_x, -z_y, 1 \rangle dx dy \\ &= \langle 1, 1, 1 \rangle dx dy \end{aligned}$$

$$\iint_S \vec{F} \cdot \hat{n} dS = \iint_S \langle x, y, z \rangle \cdot \langle 1, 1, 1 \rangle dx dy$$

$$= \iint_S (x+y+z) dx dy$$

$$= \int_0^1 \int_0^{1-x} 1 dy dx.$$

(wing  $z = 1 - x - y$ )

Inner:  $1 - x$ .

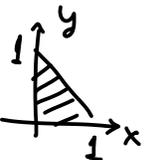
$$\text{Outer: } \int_0^1 (1-x) dx = \left[ x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}.$$

c) (10) Calculate the total flux of  $\vec{F}$  out of the tetrahedron by using the divergence theorem.

$$\text{div } \vec{F} = 1 + 1 + 1 = 3, \text{ so}$$

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_R 3 dV = 3 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx$$

$$= 3 \int_0^1 \int_0^{1-x} (1-x-y) dy dx$$



$$\text{Inner: } 3 \left[ (1-x)y - \frac{y^2}{2} \right]_0^{1-x} = \frac{3}{2} (1-x)^2$$

$$\text{Outer: } \int_0^1 \frac{3}{2} (1-x)^2 dx = \left[ -\frac{1}{2} (1-x)^3 \right]_0^1 = \frac{1}{2}.$$

(or:  $3 \text{ vol (tetrahedron)}$   
 $= 3 \cdot \left( \frac{1}{3} \cdot \text{base} \cdot \text{height} \right)$   
 $= 3 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}$ ).