Math 53 Midterm 1 Monday, October 2, 2017 2:10 - 3:00

Directions: Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. **No books, notes, calculators, or electronic devices**. Please stop when asked to and don't talk until your paper is handed in.

YOUR NAME: SOLUTIONS

Last 4 digits of your student ID: ________

Please mark the box next to your GSI's name, and circle your discussion section.

Discussion section:	☐ Nima MOINI	# 101 (8am),	# 103 (9am)
	☐ Ritwik GHOSH	# 102 (8am),	# 104 (9am)
	☐ Kyeonsik NAM	# 105 (10am),	# 108 (11am)
	☐ Jasper DENG	# 106 (10am),	# 107 (11am)
	☐ Michael YEH	# 109 (12pm),	# 111 (1pm)
	☐ Jingyi WANG	# 110 (12pm),	# 112 (1pm)
	☐ Jeffrey HICKS	# 113 (3pm),	# 115 (4pm)
	☐ Justin BRERETON	# 114 (3pm),	# 116 (5pm)
	☐ Ahmad ZAREEI	# 118 (4pm),	# 117 (5pm)

GRADING

1. ______	/23
2. ______	/17
3. ______	/15
4. ______	/20
5. ______	/25
TOTAL	/100

Problem 1. (23)

a) (8) Find an equation of the plane $\mathcal{P}$ through the points $A = (1, 0, 3)$, $B = (3, 0, 4)$, and $C = (2, 1, 4)$.

$\overrightarrow{AB} = \langle 2, 0, 1 \rangle$ $\quad$ $\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \langle -1, -1, 2 \rangle$

$\overrightarrow{AC} = \langle 1, 1, 1 \rangle$

$-x - y + 2z = 5$

(or $x + y - 2z = -5$).

b) (8) Let $\mathcal{L}$ be the line through $Q = (4, 3, 0)$ and perpendicular to the plane $\mathcal{P}$ of part (a). Find the point R where $\mathcal{L}$ intersects $\mathcal{P}$.

$\mathcal{L} \parallel \vec{N} = \langle -1, -1, 2 \rangle$, parametric equations $\begin{cases} x = 4 - t \\ y = 3 - t \\ z = 2t \end{cases}$

intersects $\mathcal{P}$ when $-x - y + 2z = (t-4) + (t-3) + 4t = 6t - 7 = 5$

$6t = 12$

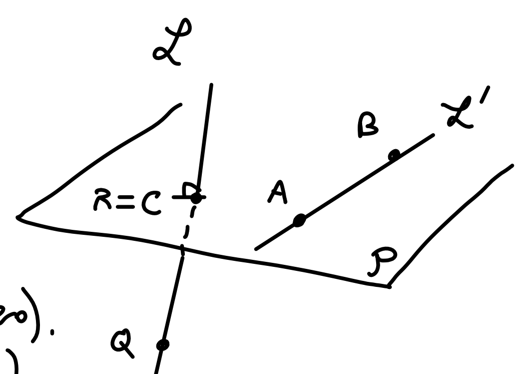
$t = 2$

so $\underline{R = (2, 1, 4)}$ $(= C)$.

c) (7) Let $\mathcal{L}$ be the same line as in part (b), and let $\mathcal{L}'$ be the line through the points A and B of part (a). Are $\mathcal{L}$ and $\mathcal{L}'$ parallel, intersecting, or skew? Justify your answer.

$\mathcal{L}, \mathcal{L}'$ are skew:

- not parallel ($\mathcal{L} \perp \mathcal{P}$, $\mathcal{L}'$ contained in $\mathcal{P}$)
- don't intersect: $\mathcal{L}'$ is contained in $\mathcal{P}$, but $\mathcal{L}$ hits $\mathcal{P}$ only at $R = C$ which is not on $\mathcal{L}'$.



(A, B, C not aligned since cross-product in (a) was non-zero).

(or: R is not on $\mathcal{L}'$ because $y = 0$ everywhere on $\mathcal{L}'$)

Problem 2. (17)

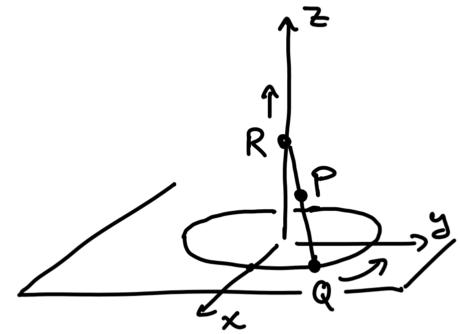
A point Q moves counterclockwise at unit speed around the unit circle in the xy-plane, starting at $(1,0,0)$ when $t=0$, while a point R moves at unit speed up the z-axis, starting at the origin when $t=0$. An elastic band connects Q to R.

a) (10) Find parametric equations for the motion of the midpoint P of the elastic band.

$Q: (\cos t, \sin t, 0)$

$R = (0,0,t)$

$\rightarrow$ midpoint $P = \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{t}{2}\right)$



b) (7) Find the length of the trajectory of P from $t=0$ to $t=10$.

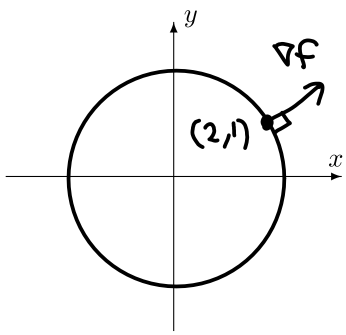
Velocity $\vec{v} = \frac{d\vec{r}}{dt} = \langle -\frac{1}{2}\sin t, \frac{1}{2}\cos t, \frac{1}{2} \rangle$

$|\vec{v}| = \sqrt{\frac{\sin^2 t}{4} + \frac{\cos^2 t}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Length $\int_0^{10} |\vec{v}|\, dt = \frac{10}{\sqrt{2}} = \underline{5\sqrt{2}}$.

Problem 3. (15) Let $f(x,y) = \ln(x^2+y^2)$.

a) (5) Sketch the level curve of f through $(x,y) = (2,1)$ and the direction of the gradient vector.



b) (5) Calculate the gradient of f at $(2,1)$.

$$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

$$\nabla f(2,1) = \left\langle \frac{4}{5}, \frac{2}{5} \right\rangle$$

c) (5) Let $x(t) = t+1$ and $y(t) = t^3$. Find $\frac{d}{dt} f(x(t), y(t))$ at $t = 1$.

At $t = 1$, $(x,y) = (2,1)$, $\frac{d\vec{r}}{dt} = \langle 1, 3t^2 \rangle = \langle 1, 3 \rangle$.

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \frac{4}{5} \cdot 1 + \frac{2}{5} \cdot 3 = \frac{10}{5} = 2.$$

Problem 4. (20) Let $f(x, y) = 3x^2y - y^3 - 6x$.

a) (8) Find all the critical points of f.

$$\begin{cases} f_x = 6xy - 6 = 0 \Leftrightarrow xy = 1 \\ f_y = 3x^2 - 3y^2 = 0 \Leftrightarrow y = \pm x \end{cases}$$

So: $\pm x^2 = 1$, $x = \pm 1$. 2 critical points $\underline{(-1,-1) \text{ and } (1,1)}$.

b) (7) Find the type of the critical point(s) which lie(s) in the first quadrant ($x \geq 0$, $y \geq 0$).

Second derivative test at $(1,1)$:

$f_{xx} = 6y = 6$

$f_{yy} = -6y = -6$ $\qquad f_{xx} f_{yy} - (f_{xy})^2 = 6 \cdot (-6) - 6^2 = -72 < 0$

$f_{xy} = 6x = 6$ $\qquad$ saddle point.

c) (5) Without any further calculation, what can you say about the maximum of f in the region where $0 \leq x \leq 5$, $0 \leq y \leq 5$?

No local max in this region, so max is achieved at a point of the boundary. (would need to test each of $x = 0$, $x = 5$, $y = 0$, $y = 5$).

Problem 5. (25) (The three parts can be solved in any order.)

a) (7) Find an equation of the tangent plane to the surface $\underbrace{x^4 + y^4 + z^4}_{g(x,y,z)} = 18$ at $(2, 1, 1)$.

Normal vector to $g = 18$:

$$\nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$$
$$= \langle 32, 4, 4 \rangle$$

$$32(x-2) + 4(y-1) + 4(z-1) = 0 \quad \text{or} \quad 32x + 4y + 4z = 72$$
$$\underline{8x + y + z = 18}$$

b) (8) The equation $x^4 + y^4 + z^4 = 18$ defines implicitly z as a function of x and y, $z = z(x, y)$. Find $\partial z/\partial x$ at $(x, y, z) = (2, 1, 1)$.

$$dg = 4x^3 dx + 4y^3 dy + 4z^3 dz = 0$$
$$\Rightarrow dz = -\frac{x^3}{z^3} dx - \frac{y^3}{z^3} dy = -8\,dx - dy.$$

So $\frac{\partial z}{\partial x} = -8$,

(or use formula $\frac{\partial z}{\partial x} = -\frac{g_x}{g_z}$, or use eq. of tangent plane $z = -8x - y + 18$ from (a)).

c) (10) Use Lagrange multipliers to find the minimum and maximum values of $f(x, y, z) = 8x + y + z$ on the surface $x^4 + y^4 + z^4 = 18$.

$\nabla f = \lambda \nabla g$: $\begin{cases} 8 = \lambda . 4x^3 \\ 1 = \lambda . 4y^3 \\ 1 = \lambda . 4z^3 \end{cases}$ so $\frac{x^3}{8} = y^3 = z^3 \left(= \frac{1}{4\lambda}\right)$.

Hence $\frac{x}{2} = y = z$

and $x^4 + y^4 + z^4 = 18 \underset{\substack{x=2z \\ y=z}}{\Rightarrow} 16z^4 + z^4 + z^4 = 18$, $z^4 = 1$, $z = \pm 1$.

So: $(x, y, z) = (2, 1, 1)$ or $(-2, -1, -1)$

$f = 18$ $f = -18$

maximum   minimum