Min-max problems continued: last time, studied critical pts

First studied in detail $w = ax^2 + bxy + cy^2$, found $4ac - b^2 > 0$ min/max, < 0 saddle

In general: 2nd derivative test:

At a critical point of $f(x,y)$, compute $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$$\text{If } D > 0 \Rightarrow \begin{cases} f_{xx} > 0 : \text{ local minimum} \\ f_{xx} < 0 : \text{ local maximum} \end{cases}$$

$D < 0 \Rightarrow$ saddle

$D = 0 \Rightarrow$ can't conclude.

- Proof: follows from quadratic approximation. (Taylor series)

$$\Delta f \approx \underbrace{\cancel{f_x \cdot (x-x_0)} + \cancel{f_y \cdot (y-y_0)}}_{0 \text{ at critical point!}} + \frac{1}{2} f_{xx} \cdot (x-x_0)^2 + f_{xy}(x-x_0)(y-y_0) + \frac{1}{2} f_{yy} (y-y_0)^2$$

$\Rightarrow$ reduce to case of a quadratic function $a(\Delta x)^2 + b\,\Delta x\,\Delta y + c(\Delta y)^2$ (studied "by hand")

with $4ac - b^2 = 4\left(\frac{1}{2} f_{xx}\right)\left(\frac{1}{2} f_{yy}\right) - (f_{xy})^2 = D$ ✓

In degenerate case ($D = 0$), can't conclude because we'd need higher order derivatives to decide! (e.g.: $x^2 + y^4$ min. vs. $x^2 - y^4$ saddle).

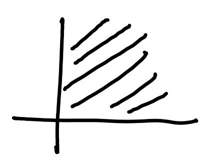
Example: $f(x,y) = x + y + \dfrac{1}{xy}$, $x, y > 0$.

$$\left.\begin{aligned} f_x &= 1 - \frac{1}{x^2 y} = 0 \\ f_y &= 1 - \frac{1}{x y^2} = 0 \end{aligned}\right\} \Rightarrow x^2 y = 1,\ x y^2 = 1 \Rightarrow x = y = 1.$$

only critical pt.

$$f_{xx} = \frac{2}{x^3 y}, \quad f_{xy} = \frac{1}{x^2 y^2}, \quad f_{yy} = \frac{2}{x y^3}$$

$D = 2 \cdot 2 - 1^2 = 3 > 0$, and $f_{xx} = 2 > 0 \Rightarrow$ local min



* But then, where's the max?

Answer: must be along boundary or at infinity!

In this example: 3 cases:
- $x \to 0$
- $y \to 0$
- $x \to \infty$ and/or $y \to \infty$.

(Since those are not in the domain strictly speaking, max won't be achieved.).

In all 3 cases, $f \to \infty$. So: min $= 3$, max not achieved (f unbounded).

To find the absolute max & min of a differentiable function on a domain D:

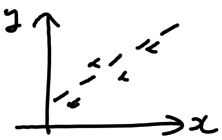
1) find the critical pts of f in D, and the values of f at these pts
2) find extreme values of f on the boundary of D and at infinity

$\Rightarrow$ the largest / smallest of all these gives the max / min

Remarks: ★ If D is closed (contains its boundary) & bounded ($|x|, |y| \le M$) then max/min are attained! Otherwise, maybe not. (cf. example above)

★ Don't actually need to determine types of crit pts! (just compare values). But can still be a good idea (to check it makes sense).

- One application of 2-variable minimization: best-fit interpolation (least-squares method).

Goal: given data points (x_1, y_1) ... (x_n, y_n) , find line $y = mx+b$ that best fits the data (useful in experimental sciences, stats, social sciences, ...)

i.e. minimize function $f(m,b) = \sum_{i=1}^{n} (y_i - (mx_i + b))^2$ (total square error).

y_i ↑ data, (mx_i+b) ↑ predicted for x_i

↳ use what we've seen!

see a future HW problem.

- So far: max/min $f(x,y)$ where variables are independent.
- Next: different problem: the variables are constrained by an equation $g(x,y) = c$ (no longer independent).

Find max/min of f subject to this constraint?

(This is very different from unconstrained min/max: want min/max of f on the level set $g = c$, don't care about values elsewhere!)

⇝ need other tools: method of Lagrange multipliers