Math 53 Homework 9 – Solutions

15.6 # 13: $\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$. Inner: $[6xyz]_0^{1+x+y} = 6xy(1+x+y)$.
Middle: $\int_0^{\sqrt{x}} 6xy + 6x^2y + 6xy^2\,dy = \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_0^{\sqrt{x}} = 3x^2 + 3x^3 + 2x^{5/2}$.
Outer: $\int_0^1 3x^2 + 3x^3 + 2x^{5/2}\,dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 = 1 + \frac{3}{4} + \frac{4}{7} = \frac{65}{28}$.

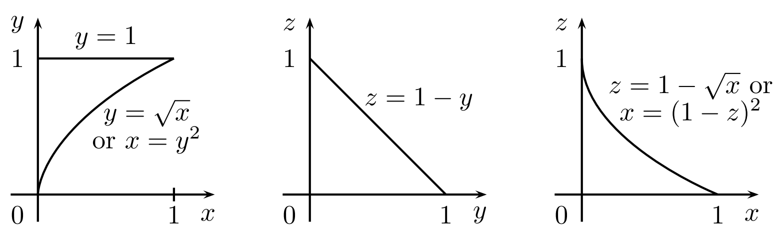
15.6 # 15: The base of the tetrahedron is the triangle with vertices (0,0),(2,0),(0,2) in the xy-plane, i.e. the region $0 \le y \le 2-x$, $0 \le x \le 2$. The top face is $x+y+z=2$, i.e. $z = 2-x-y$, while the bottom face is $z=0$. So $\iiint_T y^2\,dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} y^2\,dz\,dy\,dx$.
Inner: $[y^2z]_0^{2-x-y} = y^2(2-x-y)$.
Middle: $\int_0^{2-x} y^2(2-x-y)\,dy = \left[\frac{1}{3}(2-x)y^3 - \frac{1}{4}y^4\right]_0^{2-x} = \frac{1}{3}(2-x)^4 - \frac{1}{4}(2-x)^4 = \frac{1}{12}(2-x)^4$.
Outer: $\int_0^2 \frac{1}{12}(2-x)^4\,dx = \left[-\frac{1}{60}(2-x)^5\right]_0^2 = 0 + \frac{2^5}{60} = \frac{32}{60}$.

15.6 # 18: The solid is a piece of the cylinder of radius 3 centered on the x-axis, between the planes $x=0$ (back) and $y=3x$ (front), and above the xy-plane. The base of this solid in the xy-plane is a triangle with vertices $(x,y) = (0,0)$, $(1,3)$, and $(0,3)$ (the rightmost edge is $y=3$ where the cylinder intersects the xy-plane). Integrating over z first:
$\iiint_E z\,dV = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx$. Inner: $[\frac{1}{2}z^2]_0^{\sqrt{9-y^2}} = \frac{1}{2}(9-y^2)$.
Middle: $\int_{3x}^3 \frac{1}{2}(9-y^2)\,dy = [\frac{9}{2}y - \frac{1}{6}y^3]_{3x}^3 = \frac{27}{2} - \frac{27}{6} - \frac{27}{2}x + \frac{27}{6}x^3 = 9 - \frac{27}{2}x + \frac{9}{2}x^3$.
Outer: $\int_0^1 9 - \frac{27}{2}x + \frac{9}{2}x^3\,dx = [9x - \frac{27}{4}x^2 + \frac{9}{8}x^4]_0^1 = 9 - \frac{27}{4} + \frac{9}{8} = \frac{27}{8}$.
Or easier, integrating over x first ($0 \le x \le y/3$; $y^2+z^2 \le 9$, $y,z \ge 0$):

$$\iiint_E z\,dV = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{y/3} z\,dx\,dz\,dy.$$

Inner: $\frac{1}{3}yz$. Middle: $[\frac{1}{6}yz^2]_0^{\sqrt{9-y^2}} = \frac{3}{2}y - \frac{1}{6}y^3$. Outer: $[\frac{3}{4}y^2 - \frac{1}{24}y^4]_0^3 = \frac{27}{8}$.

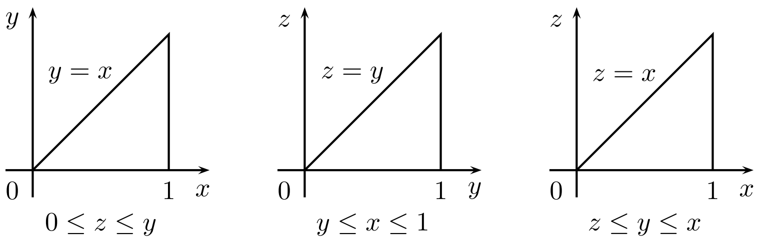
15.6 # 33: The projections of E on the coordinate planes are:

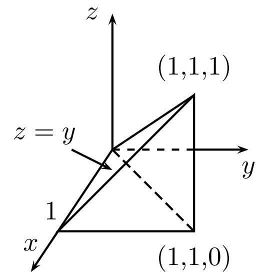


$0 \le z \le 1-y$ $\qquad$ $0 \le x \le y^2$ $\qquad$ $\sqrt{x} \le y \le 1-z$

$$\begin{aligned}
&\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &&= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,dz\,dx\,dy \\
= &\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz &&= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy \\
= &\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx &&= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

15.6 # 35: The region and its projections are:





$$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy \quad = \int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx \quad = \int_0^1\int_0^y\int_y^1 f(x,y,z)\,dx\,dz\,dy$$
$$= \int_0^1\int_z^1\int_y^1 f(x,y,z)\,dx\,dy\,dz \quad = \int_0^1\int_0^x\int_z^x f(x,y,z)\,dy\,dz\,dx \quad = \int_0^1\int_z^1\int_z^x f(x,y,z)\,dy\,dx\,dz$$

15.6 # 46: $I_z = \iiint_E (x^2+y^2)\,\rho\,dV = \int_{-h}^{h}\int_{-\sqrt{h^2-x^2}}^{\sqrt{h^2-x^2}}\int_{\sqrt{x^2+y^2}}^{h}(x^2+y^2)\,k\,dz\,dy\,dx.$

Or better, in cylindrical coordinates: $I_z = \int_0^{2\pi}\int_0^h\int_r^h r^2\,k\,dz\,r\,dr\,d\theta.$

Inner: $[r^2kz]_r^h = khr^2 - kr^3$. Middle: $\int_0^h (khr^3 - kr^4)\,dr = \left[\frac{1}{4}khr^4 - \frac{1}{5}kr^5\right]_0^h = \frac{1}{20}kh^5.$

Outer: $2\pi\cdot\frac{1}{20}kh^5 = \frac{\pi}{10}kh^5.$

15.6 # 54: Volume $= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} dz\,r\,dr\,d\theta = \frac{2}{3}\pi$ (half of unit sphere)

Average value of height (=z): $\bar{z} = \frac{1}{2\pi/3}\iiint_E z\,dV = \frac{3}{2\pi}\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} z\,dz\,r\,dr\,d\theta.$

Inner: $\left[\frac{1}{2}z^2\right]_0^{\sqrt{1-r^2}} = \frac{1}{2}(1-r^2).$

Middle: $\int_0^1 \frac{1}{2}(1-r^2)\,r\,dr = \left[\frac{1}{4}r^2 - \frac{1}{8}r^4\right]_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$

Outer: $\bar{z} = \frac{3}{2\pi}(2\pi)\frac{1}{8} = \frac{3}{8}.$

15.7 # 21: The equation of the cone is $z^2 = 4x^2 + 4y^2 = 4r^2$ or, for the portion above the xy-plane, $z = 2r$. So E is the solid region where $0 \le z \le 2r$ and $r \le 1$.

$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r}(r\cos\theta)^2 r\,dz\,dr\,d\theta.$

Inner: $\int_0^{2r}(r\cos\theta)^2 r\,dz = 2r^4\cos^2\theta.$ Middle: $\int_0^1 2r^4\cos^2\theta\,dr = \frac{2}{5}\cos^2\theta.$

Outer: $\int_0^{2\pi}\frac{2}{5}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{1}{5}(1+\cos 2\theta)\,d\theta = \left[\frac{1}{5}\theta + \frac{1}{10}\sin 2\theta\right]_0^{2\pi} = 2\pi/5.$

15.7 # 22: E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$. So its volume is $\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta = 2\pi\int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \frac{4}{3}\pi(8-3\sqrt{3}).$

15.7 # 30: the bounds for z are $0 \le z \le 9 - x^2 - y^2 = 9 - r^2$, and those for (x,y) are $0 \le y \le \sqrt{9-x^2}$, or $x^2 + y^2 \le 9$, $y \ge 0$: half of a disc of radius 3 (corresponding to $0 \le \theta \le \pi$, $r \le 3$). So: $\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_0^{\pi}\int_0^3\int_0^{9-r^2} r\,r\,dz\,dr\,d\theta.$

Inner: $\int_0^{9-r^2} r^2\,dz = r^2(9-r^2) = 9r^2 - r^4.$

Middle: $\int_0^3 9r^2 - r^4\,dr = \left[3r^3 - \frac{1}{5}r^5\right]_0^3 = 162/5.$

Outer: $162\pi/5.$

Problem 1: $M = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} dxdydz = \int_0^1 (1-z^2)dz = 2/3.$

$$\bar{z} = \frac{1}{M}\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} zdxdydz = \frac{3}{2}\int_0^1 (z-z^3)dz = \frac{3}{8}.$$

$$\bar{x} = \frac{1}{M}\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} xdxdydz = \frac{3}{2}\int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1-z^2}{2}dydz$$
$$= \frac{3}{4}\int_0^1 (1-z^2)^{3/2}dz = \frac{3}{4}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3}{4}\frac{3\pi}{16} = \frac{9\pi}{64} \qquad (z = \sin\theta,\ dz = \cos\theta\, d\theta)$$

using double angle formulas twice to calculate

$$\int_0^{\pi/2} \cos^4\theta\, d\theta = \int_0^{\pi/2} \tfrac{1}{4}(1+\cos 2\theta)^2\, d\theta = \int_0^{\pi/2} (\tfrac{3}{8} + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{8}\cos 4\theta)\, d\theta = \tfrac{3\pi}{16}.$$

By symmetry with respect to the plane $x = y$, $\bar{x} = \bar{y}$. Thus the centroid is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{9\pi}{64}, \frac{9\pi}{64}, \frac{3}{8}\right).$$

15.8 # 14: $\rho \le \csc\phi \Leftrightarrow \rho\sin\phi \le 1 \Leftrightarrow r \le 1$, or equivalently $x^2+y^2 \le 1$, which corresponds to the solid cylinder of unit radius centered on the z-axis. Moreover, $\rho \le 2$ corresponds to the solid sphere of radius 2 centered at the origin. Hence, this solid is the portion of the solid cylinder $x^2 + y^2 \le 1$ that lies inside the sphere of radius 2.
(The sphere and the cylinder intersect at the two circles $r = 1$, $z = \pm\sqrt{3}$; so the boundary of the solid is given by the portion of the cylinder where $-\sqrt{3} \le z \le \sqrt{3}$, and spherical caps at the top and bottom).

15.8 # 15: The cone $z = \sqrt{x^2+y^2}$ corresponds to $z = r$, i.e. $\rho\cos\phi = \rho\sin\phi$, i.e. $\phi = \pi/4$. (See also 15.8 # 9(a)). Thus, the region above the cone corresponds to $\phi \le \pi/4$.
In spherical coordinates, the sphere $x^2 + y^2 + z^2 = z$ (centered at $(0, 0, \frac{1}{2})$ and of radius $\frac{1}{2}$, since the equation rewrites as $x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$) has equation $\rho = \cos\phi$. (This can be seen either geometrically on a slice by a vertical plane, or by manipulating the equation: $x^2 + y^2 + z^2 = z$ becomes $\rho^2 = \rho\cos\phi$, which simplifies to $\rho = \cos\phi$).
Hence, the solid is described by the inequalities $\rho \le \cos\phi$, $0 \le \phi \le \pi/4$.
(See Example 4 on page 1048 [7th ed: p. 1036] for more details.)

15.8 # 19: In cylindrical coordinates: $0 \le z \le 2$, $0 \le r \le 3$, $0 \le \theta \le \pi/2$, so the integral is given by $\int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$.
In spherical coordinates: the top plane has equation $z = \rho\cos\phi = 2$, i.e. $\rho = 2\sec\phi$. The cylinder corresponds to $r = \rho\sin\phi = 3$, i.e. $\rho = 3\csc\phi$. They intersect when $2\sec\phi = 3\csc\phi$, i.e. $\tan\phi = 3/2$. Therefore:

$$\int_0^{\pi/2}\int_0^{\tan^{-1}(3/2)}\int_0^{2\sec\phi} f\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{\pi/2}\int_{\tan^{-1}(3/2)}^{\pi/2}\int_0^{3\csc\phi} f\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

15.8 # 26: E corresponds to $\phi \le \pi/4$ and $1 \le \rho \le 2$.
So: $\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_1^2 \rho\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.
Inner: $\int_1^2 \rho^3 \sin\phi\, d\rho = [\frac{1}{4}\rho^4 \sin\phi]_1^2 = \frac{15}{4}\sin\phi$.
Middle: $\frac{15}{4}\int_0^{\pi/4} \sin\phi\, d\phi = \frac{15}{4}[-\cos\phi]_0^{\pi/4} = \frac{15}{4}\left(1 - \frac{1}{\sqrt{2}}\right)$. Outer: $\frac{15\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$.

15.8 # 30: In spherical coordinates, the region below the cone $z = \sqrt{x^2+y^2}$ and above the xy-plane corresponds to $\pi/4 \le \phi \le \pi/2$. Therefore $\iiint_E dV = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.
Inner: $\left[\frac{1}{3}\rho^3 \sin\phi\right]_0^2 = \frac{8}{3}\sin\phi$. Middle: $\left[-\frac{8}{3}\cos\phi\right]_{\pi/2}^{\pi/4} = \frac{8}{3}\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}}{3}$. Outer: $2\pi \cdot \frac{4\sqrt{2}}{3} = \frac{8\pi\sqrt{2}}{3}$.

15.8 # 33: We take the hemisphere to be the region lying above the xy-plane and inside the sphere $x^2+y^2+z^2 = a^2$; and denote by K its (constant) density. So the base is contained in the xy-plane.
(a) By symmetry, the centroid lies on the z-axis, so we only need compute $\bar{z}$. Also, the mass of the hemisphere is $K \cdot (\text{volume}) = \frac{2}{3}K\pi a^3$. Therefore:
$\bar{z} = \frac{1}{\text{mass}} \iiint z\, K\, dV = \frac{3}{2K\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (\rho\cos\phi)\, K\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.
Inner: $\left[\frac{1}{4}K\rho^4 \cos\phi \sin\phi\right]_0^a = \frac{1}{4}Ka^4 \cos\phi\sin\phi$.
Middle: $\frac{1}{4}Ka^4 \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = \frac{1}{4}Ka^4 \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} = \frac{1}{8}Ka^4$.
Outer: $\frac{3}{2K\pi a^3}\,(2\pi)\,(\frac{1}{8}Ka^4) = \frac{3}{8}a$. So the centroid is $(0, 0, \frac{3}{8}a)$.

(b) We use the same setup as before, and compute the moment of inertia about the x-axis, I_x. (One could also compute I_y instead; by symmetry $I_x = I_y$).
$I_x = \iiint (y^2+z^2)\, K\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (\rho^2 \sin^2\phi \sin^2\theta + \rho^2\cos^2\phi)\, K\, \rho^2 \sin\phi\, d\rho d\phi d\theta$.
Inner: $\frac{1}{5}Ka^5(\sin^3\phi\sin^2\theta + \cos^2\phi\sin\phi)$.
Middle: $\frac{1}{5}Ka^5 \int_0^{\pi/2} \sin^2\theta(1-\cos^2\phi)\sin\phi + \cos^2\phi\sin\phi\, d\phi =$
$$= \frac{1}{5}Ka^5 \left[\sin^2\theta(\tfrac{1}{3}\cos^3\phi - \cos\phi) - \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2} = \frac{1}{5}Ka^5(\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}).$$
Outer: $I_x = \frac{1}{5}Ka^5 \int_0^{2\pi} (\frac{2}{3}\sin^2\theta + \frac{1}{3})\, d\theta = \frac{1}{5}Ka^5 \int_0^{2\pi} (\frac{2}{3} - \frac{1}{3}\cos 2\theta)\, d\theta = \frac{4}{15}\pi K a^5$.
Note: a more efficient setup for this calculation would have been to instead take the hemisphere to be the "right" half of the solid sphere $x^2+y^2+z^2 \le a^2$, i.e. where $y \ge 0$. The bounds are then $\rho \le a$, $0 \le \theta \le \pi$. Since the base is now a disk in the xz-plane, we can now compute the moment of inertia about the z-axis:
$I_z = \int_0^{\pi} \int_0^{\pi} \int_0^a (\rho^2\sin^2\phi)\, K\, (\rho^2\sin\phi)\, d\rho d\phi d\theta = \cdots = \frac{4}{15}\pi K a^5$.

15.8 # 35: In spherical coordinates $z = \sqrt{x^2+y^2}$ becomes $\phi = \pi/4$. So the volume is $V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi\, d\rho d\phi d\theta = (\int_0^{2\pi} d\theta)(\int_0^{\pi/4} \sin\phi\, d\phi)(\int_0^1 \rho^2 d\rho) = \frac{2\pi}{3}[-\cos\phi]_0^{\pi/4} = \frac{\pi(2-\sqrt{2})}{3}$.
By symmetry the centroid is on the z-axis, i.e. $\bar{x} = \bar{y} = 0$, and $\bar{z} = \frac{1}{V}\iiint z\, dV$, so
$\bar{z} = \frac{3}{\pi(2-\sqrt{2})} \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho\cos\phi\, \rho^2 \sin\phi\, d\rho d\phi d\theta = \frac{3}{\pi(2-\sqrt{2})}(2\pi)(\int_0^{\pi/4} \sin\phi\cos\phi\, d\phi)(\int_0^1 \rho^3\, d\rho)$
$= \frac{3}{4(2-\sqrt{2})}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/4} = \frac{3}{8(2-\sqrt{2})}$.

Problem 2: In cylindrical coordinates, distance to the origin is $d = \sqrt{r^2 + z^2}$, and

$$\begin{aligned}
\bar{d} &= \frac{1}{4\pi a^3/3} \int_0^{2a} \int_0^{2\pi} \int_0^{\sqrt{a^2-(z-a)^2}} \sqrt{r^2+z^2}\, r\, dr d\theta dz \\
&= \frac{3}{4\pi a^3} \int_0^{2a} \int_0^{2\pi} \frac{1}{3}(r^2+z^2)^{3/2} \Big|_{r=0}^{r=\sqrt{a^2-(z-a)^2}} d\theta dz \\
&= \frac{1}{4\pi a^3} \int_0^{2a} \int_0^{2\pi} ((2az)^{3/2} - z^3)\, d\theta dz \\
&= \frac{1}{2a^3} \int_0^{2a} ((2az)^{3/2} - z^3)\, dz \\
&= \frac{1}{2a^3} \left(\frac{2}{5}(2a)^4 - \frac{1}{4}(2a)^4 \right) = \frac{6}{5}a.
\end{aligned}$$

In spherical coordinates, $d = \rho$, $dV = \rho^2 \sin\phi\, d\rho d\phi d\theta$,

$$\begin{aligned}
\bar{d} &= \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2a\cos\phi} \rho\, \rho^2 \sin\phi\, d\rho d\phi d\theta \\
&= \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{4}(2a\cos\phi)^4 \sin\phi\, d\phi d\theta \\
&= \frac{3}{4\pi a^3} \int_0^{2\pi} \frac{1}{4}(2a)^4 \frac{-1}{5}(\cos\phi)^5 \Big|_0^{\pi/2} d\theta \\
&= \frac{3}{4\pi a^3} \frac{8\pi a^4}{5} = \frac{6a}{5}.
\end{aligned}$$