

Math 53 Homework 8 – Solutions

15.3 # 11: D is the right half of the circle of radius 2 centered at the origin. So

$$\iint_D e^{-x^2-y^2} dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta. \text{ Inner: } \left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \frac{1}{2}(1-e^{-4}). \text{ Outer: } \frac{\pi}{2}(1-e^{-4}).$$

15.3 # 13: R is the portion of the annulus $1 \leq r \leq 2$ in the first quadrant below the line $y = x$, i.e. $0 \leq \theta \leq \pi/4$. Also remember that $\arctan(y/x) = \theta$. So

$$\iint_R \arctan(y/x) dA = \int_0^{\pi/4} \int_1^2 \theta r dr d\theta = \int_0^{\pi/4} \left[\frac{1}{2}\theta r^2\right]_1^2 d\theta = \int_0^{\pi/4} \frac{3}{2}\theta d\theta = \left[\frac{3}{4}\theta^2\right]_0^{\pi/4} = \frac{3\pi^2}{64}.$$

15.3 # 15: One loop corresponds to a maximal interval of θ values for which $r = \cos 3\theta \geq 0$, for example $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Thus the area is

$$\int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r dr d\theta = \int_{-\pi/6}^{\pi/6} \frac{1}{2} \cos^2 3\theta d\theta = \int_{-\pi/6}^{\pi/6} \frac{1}{4} (1 + \cos 6\theta) d\theta = \left[\frac{1}{4} + \frac{1}{24} \sin 6\theta\right]_{-\pi/6}^{\pi/6} = \frac{\pi}{12}.$$

15.3 # 25: The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 1$, or $x^2 + y^2 = \frac{1}{2}$. So

$$\begin{aligned} V &= \iint_{x^2+y^2 \leq 1/2} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r dr d\theta = \\ &= 2\pi \int_0^{1/\sqrt{2}} (r\sqrt{1-r^2} - r^2) dr = 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3\right]_0^{1/\sqrt{2}} = \frac{\pi}{3}(2 - \sqrt{2}). \end{aligned}$$

15.3 # 31: $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dx dy$: the region of integration is the portion of the disk $x^2 + y^2 \leq 1$ where $x \geq \sqrt{3}y$ and $y \geq 0$, i.e. between the x -axis and the line $y = x/\sqrt{3}$.

$$\text{So } \int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dx dy = \int_0^{\pi/6} \int_0^1 (r \cos \theta)(r \sin \theta)^2 r dr d\theta = \int_0^{\pi/6} \int_0^1 r^4 \sin^2 \theta \cos \theta dr d\theta.$$

$$\text{Inner: } \frac{1}{5} r^5 \sin^2 \theta \cos \theta \Big|_0^1 = \frac{1}{5} \sin^2 \theta \cos \theta.$$

$$\text{Outer: } \frac{1}{5} \int_0^{\pi/6} \sin^2 \theta \cos \theta d\theta = \frac{1}{15} \sin^3 \theta \Big|_0^{\pi/6} = \frac{1}{120}.$$

15.3 # 40: a) Using polar coordinates: $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta =$
 $= 2\pi \int_0^\infty r e^{-r^2} dr = 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = 2\pi(0 + \frac{1}{2}) = \pi.$

Or, to be more rigorous, we integrate over the disk D_a and take the limit as $a \rightarrow \infty$:

$$\iint_{D_a} e^{-(x^2+y^2)} dA = \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta = 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi(1 - e^{-a^2}).$$

The result then follows, since $\lim_{a \rightarrow \infty} \pi(1 - e^{-a^2}) = \pi.$

b) $\iint_{S_a} e^{-(x^2+y^2)} dA = \int_{-a}^a \int_{-a}^a e^{-x^2} e^{-y^2} dy dx = \left(\int_{-a}^a e^{-x^2} dx\right) \left(\int_{-a}^a e^{-y^2} dy\right).$

Taking the limit as $a \rightarrow \infty$ on both sides, we deduce:

$$\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \left(\int_{-\infty}^\infty e^{-x^2} dx\right) \left(\int_{-\infty}^\infty e^{-y^2} dy\right).$$

c) Since $\int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-x^2} dx$ (the name of the integration variable is irrelevant), the result of (b) becomes: $\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi$. Taking the square root (and observing that $e^{-x^2} > 0$ for all x so the integral is positive), we get: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

d) Letting $x = t/\sqrt{2}$, we have $\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-t^2/2} dt$. Hence, $\sqrt{\pi} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2/2} dt$. Equivalently, $\int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}$.

$$\begin{aligned} \mathbf{15.4 \# 10:} \quad m &= \iint_D \rho dA = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y dy dx = \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}y^2\right]_0^{\cos x} dx = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x dx = \\ &= \int_{-\pi/2}^{\pi/2} \frac{1}{4}(1 + \cos 2x) dx = \left[\frac{1}{4}x + \frac{1}{8} \sin 2x\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{4}. \end{aligned}$$

$\bar{x} = \frac{1}{m} \iint_D x \rho dA = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} xy dy dx = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}xy^2\right]_0^{\cos x} dx = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \cos^2 x dx = 0$ by parity ($x \cos^2 x$ is an odd function). In fact, the symmetry of D and of ρ about the y -axis implies readily that $\bar{x} = 0$.

$$\begin{aligned} \bar{y} &= \frac{1}{m} \iint_D y \rho dA = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y^2 dy dx = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{1}{3}y^3\right]_0^{\cos x} dx = \frac{4}{3\pi} \int_{-\pi/2}^{\pi/2} \cos^3 x dx = \\ &= \frac{4}{3\pi} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x) \cos x dx = \frac{4}{3\pi} \left[\sin x - \frac{1}{3} \sin^3 x\right]_{-\pi/2}^{\pi/2} = \frac{4}{3\pi} \left(\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right) = \frac{16}{9\pi}. \end{aligned}$$

$$\mathbf{15.4 \# 12:} \quad \rho(x, y) = k(x^2 + y^2) = kr^2, \quad m = \int_0^{\pi/2} \int_0^1 kr^2 r dr d\theta = \frac{\pi}{2} \int_0^1 kr^3 dr = \frac{\pi}{8}k.$$

$$\begin{aligned} \bar{x} &= \frac{1}{m} \iint x \rho dA = \frac{1}{m} \int_0^{\pi/2} \int_0^1 (r \cos \theta) (kr^2) r dr d\theta = \frac{1}{m} \int_0^{\pi/2} \left[\frac{1}{5}kr^5 \cos \theta\right]_{r=0}^{r=1} d\theta = \\ &= \frac{1}{m} \frac{k}{5} \int_0^{\pi/2} \cos \theta d\theta = \frac{k/5}{k\pi/8} [\sin \theta]_0^{\pi/2} = \frac{8}{5\pi}. \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m} \iint y \rho dA = \frac{1}{m} \int_0^{\pi/2} \int_0^1 (r \sin \theta) (kr^2) r dr d\theta = \frac{1}{m} \int_0^{\pi/2} \left[\frac{1}{5}kr^5 \sin \theta\right]_{r=0}^{r=1} d\theta = \\ &= \frac{1}{m} \frac{k}{5} \int_0^{\pi/2} \sin \theta d\theta = \frac{k/5}{k\pi/8} [-\cos \theta]_0^{\pi/2} = \frac{8}{5\pi}. \end{aligned}$$

(Note: $\bar{x} = \bar{y}$ by symmetry: the lamina and the density are symmetric about $y = x$.)

$$\begin{aligned} \text{Moments of inertia: } I_x &= \iint_D y^2 \rho dA = \int_0^{\pi/2} \int_0^1 (r^2 \sin^2 \theta) (kr^2) r dr d\theta = \int_0^{\pi/2} \left[\frac{1}{6}kr^6 \sin^2 \theta\right]_0^1 d\theta \\ &= \frac{1}{6}k \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{6}k \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta) d\theta = \frac{1}{6}k \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{24}k. \end{aligned}$$

$$\begin{aligned} I_y &= \iint_D x^2 \rho dA = \int_0^{\pi/2} \int_0^1 (r^2 \cos^2 \theta) (kr^2) r dr d\theta = \int_0^{\pi/2} \left[\frac{1}{6}kr^6 \cos^2 \theta\right]_0^1 d\theta = \frac{1}{6}k \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \frac{1}{6}k \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{1}{6}k \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{24}k \quad (= I_x \text{ by symmetry}). \end{aligned}$$

$$I_0 = \iint_D r^2 \rho dA = \int_0^{\pi/2} \int_0^1 r^2 (kr^2) r dr d\theta = \frac{\pi}{2} \left[\frac{1}{6}kr^6\right]_0^1 = \frac{\pi}{12}k.$$

(Or: recall $I_0 = I_x + I_y$, so using symmetry $I_x = I_y = \frac{1}{2}I_0$.)

15.4 # 28: a) $f(x, y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Here $f(x, y) = 0$ outside of the unit square, so we just need to compute $\int_0^1 \int_0^1 f(x, y) dy dx = \int_0^1 \int_0^1 4xy dy dx = \int_0^1 [2xy^2]_{y=0}^{y=1} dx = \int_0^1 2x dx = [x^2]_0^1 = 1$.

b) (i) The region where $x \geq \frac{1}{2}$ corresponds to the right half of the unit square (recall that X and Y only take values between 0 and 1).

$$\text{So } P(X \geq \frac{1}{2}) = \int_{1/2}^1 \int_0^1 4xy dy dx = \int_{1/2}^1 [2xy^2]_{y=0}^{y=1} dx = \int_{1/2}^1 2x dx = [x^2]_{1/2}^1 = \frac{3}{4}.$$

$$\text{(ii) } P(X \geq \frac{1}{2}, Y \leq \frac{1}{2}) = \int_{1/2}^1 \int_0^{1/2} 4xy dy dx = \int_{1/2}^1 [2xy^2]_0^{1/2} dx = \int_{1/2}^1 \frac{x}{2} dx = \left[\frac{x^2}{4}\right]_{1/2}^1 = \frac{3}{16}.$$

$$\text{c) } E(X) = \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^1 \int_0^1 x(4xy) dy dx = \int_0^1 [2x^2y^2]_0^1 dx = \int_0^1 2x^2 dx = \frac{2}{3}.$$

$$E(Y) = \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^1 \int_0^1 y(4xy) dy dx = \int_0^1 \left[\frac{4}{3}xy^3\right]_0^1 dx = \int_0^1 \frac{4}{3}x dx = \frac{2}{3}.$$

Problem 1:

a) Average distance = $\frac{1}{\text{Area}} \iint r \, dA = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r \, r \, dr \, d\theta = \frac{1}{\pi a^2} 2\pi \left[\frac{1}{3} r^3 \right]_0^a = \frac{2}{3} a.$

b) (Using the setup suggested by the hint, so the circle has polar equation $r = 2a \cos \theta$):

$$\begin{aligned} \text{Average distance} &= \frac{1}{\text{Area}} \iint_r dA = \frac{1}{\pi a^2} \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} r^2 \, dr \, d\theta = \frac{1}{\pi a^2} \int_{-\pi/2}^{\pi/2} \frac{1}{3} (2a \cos \theta)^3 \, d\theta \\ &= \frac{8a^3}{3\pi a^2} \int_{-\pi/2}^{\pi/2} \cos^3 \theta \, d\theta = \frac{8a}{3\pi} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta) \cos \theta \, d\theta = (\text{substituting } u = \sin \theta) \\ &= \frac{8a}{3\pi} \int_{-1}^1 (1 - u^2) \, du = \frac{8a}{3\pi} \left[u - \frac{1}{3} u^3 \right]_{-1}^1 = \frac{8a}{3\pi} \left(\frac{2}{3} - \left(-\frac{2}{3}\right) \right) = \frac{32a}{9\pi}. \end{aligned}$$

Problem 2:

a) Area = $\int_0^{\pi/2} \int_0^{\sin 2\theta} r \, dr \, d\theta.$ Inner: $\left[\frac{1}{2} r^2 \right]_0^{\sin 2\theta} = \frac{1}{2} \sin^2 2\theta.$

Outer: $\frac{1}{2} \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta = \frac{1}{4} \theta - \frac{1}{16} \sin 4\theta \Big|_0^{\pi/2} = \frac{\pi}{8}.$

b) By symmetry the centroid must be on the diagonal line $y = x$, so calculating \bar{x} is enough.

$$\bar{x} = \frac{1}{\text{Area}} \int_0^{\pi/2} \int_0^{\sin 2\theta} r \cos \theta \, r \, dr \, d\theta.$$

Inner: $\left[\frac{1}{3} r^3 \cos \theta \right]_0^{\sin 2\theta} = \frac{1}{3} \sin^3 2\theta \cos \theta = \frac{1}{3} (2 \sin \theta \cos \theta)^3 \cos \theta = \frac{8}{3} \sin^3 \theta \cos^4 \theta$
 $= \frac{8}{3} \sin \theta \cos^4 \theta (1 - \cos^2 \theta) = \frac{8}{3} \sin \theta (\cos^4 \theta - \cos^6 \theta).$

Outer: $\frac{8}{3} \left(-\frac{1}{5} \cos^5 \theta + \frac{1}{7} \cos^7 \theta \right) \Big|_0^{\pi/2} = \frac{8}{3} \left(\frac{1}{5} - \frac{1}{7} \right) = \frac{16}{105}.$ Therefore $\bar{x} = \bar{y} = \frac{8}{\pi} \frac{16}{105} = \frac{128}{105\pi}.$

15.9 # 15: $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3,$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v.$

To find the region S in the uv -plane that corresponds to R , we first find how the boundary maps under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$; so it corresponds to $u + 2v = \frac{1}{2}(2u + v)$, which simplifies to $v = 0$. The line through $(0,0)$ and $(1,2)$ is $y = 2x$; this corresponds to $u + 2v = 2(2u + v)$, which simplifies to $u = 0$. Finally, the line through $(1,2)$ and $(2,1)$ is $x + y = 3$, which becomes $(2u + v) + (u + 2v) = 3$, which simplifies to $u + v = 1$. So S is the triangle in the uv -plane bounded by the lines $u = 0$, $v = 0$, and $u + v = 1$ (i.e. $v = 1 - u$).

Therefore $\iint_R (x - 3y) \, dA = \int_0^1 \int_0^{1-u} (-u - 5v) |3| \, dv \, du = -3 \int_0^1 \int_0^{1-u} (u + 5v) \, dv \, du.$

Inner: $\int_0^{1-u} (u + 5v) \, dv = \left[uv + \frac{5}{2} v^2 \right]_0^{1-u} = u(1-u) + \frac{5}{2} (1-u)^2 = \frac{3}{2} u^2 - 4u + \frac{5}{2}.$

Outer: $-3 \int_0^1 \left(\frac{3}{2} u^2 - 4u + \frac{5}{2} \right) \, du = -3 \left[\frac{1}{2} u^3 - 2u^2 + \frac{5}{2} u \right]_0^1 = -3.$

15.9 # 19: $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$ The integrand is $xy = u.$

Since $xy = u$, the hyperbolas $xy = 1$ and $xy = 3$ correspond to the lines $u = 1$ and $u = 3$ respectively. Moreover $y = x \Leftrightarrow v = \frac{u}{v} \Leftrightarrow v^2 = u$, and $y = 3x \Leftrightarrow v = 3\frac{u}{v} \Leftrightarrow v^2 = 3u$.

Since we are in the first quadrant, $y \geq 0$ so $v \geq 0$. Hence the region of integration corresponds to $\sqrt{u} \leq v \leq \sqrt{3u}$, $1 \leq u \leq 3$. Thus

$$\begin{aligned} \iint_R xy \, dA &= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \frac{1}{v} \, dv \, du = \int_1^3 u \left(\ln \sqrt{3u} - \ln \sqrt{u} \right) \, du = \int_1^3 u \ln \sqrt{3} \, du = \\ &= \left[\frac{1}{2} u^2 \right]_1^3 \ln \sqrt{3} = 4 \ln \sqrt{3} = 2 \ln 3. \end{aligned}$$

15.9 # 23: Let $u = x - 2y$ and $v = 3x - y$: then $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$, so $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{5}$.

With this change of variables, R is the image of the rectangle $0 \leq u \leq 4$, $1 \leq v \leq 8$. So

$$\iint_R \frac{x - 2y}{3x - y} \, dA = \int_0^4 \int_1^8 \frac{u}{v} \frac{1}{5} \, dv \, du = \frac{1}{5} \left(\int_0^4 u \, du \right) \left(\int_1^8 \frac{dv}{v} \right) = \frac{1}{5} \left[\frac{u^2}{2} \right]_0^4 \left[\ln v \right]_1^8 = \frac{8}{5} \ln 8.$$

Problem 3: $u = xy$, $v = y/x$: so $uv = y^2$ and $u/v = x^2$, which gives $x^2 + y^2 = uv + u/v$.

The Jacobian is $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v$.

Thus $du \, dv = \left| \frac{2y}{x} \right| dx \, dy$, and $dx \, dy = \left| \frac{x}{2y} \right| du \, dv = \frac{1}{2|v|} du \, dv$.

Limits of integration: $0 \leq xy \leq 1$, $1 \leq x \leq 2$. In uv -coordinates, the first inequality becomes $0 \leq u \leq 1$; and the second one becomes $1 \leq x^2 = u/v \leq 4$, or equivalently $v \leq u \leq 4v$, which means that $v \leq u$ and $v \geq \frac{1}{4}u$. So

$$\begin{aligned} \iint_R (x^2 + y^2) \, dx \, dy &= \int_0^1 \int_{u/4}^u \left(uv + \frac{u}{v} \right) \frac{1}{2v} \, dv \, du = \int_0^1 \int_{u/4}^u \left(\frac{u}{2} + \frac{u}{2v^2} \right) \, dv \, du \\ &= \int_0^1 \left[\frac{uv}{2} - \frac{u}{2v} \right]_{u/4}^u \, du = \int_0^1 \left(\left(\frac{1}{2}u^2 - \frac{1}{2} \right) - \left(\frac{1}{8}u^2 - 2 \right) \right) \, du \\ &= \int_0^1 \left(\frac{3}{8}u^2 + \frac{3}{2} \right) \, du = \left[\frac{1}{8}u^3 + \frac{3}{2}u \right]_0^1 = \frac{1}{8} + \frac{3}{2} = \frac{13}{8}. \end{aligned}$$