Math 53 Homework 7 – Solutions

Problem 1. a) The variables are m and b, and we are trying to minimize $f(m,b) = \sum_{i=1}^{n} (y_i - (mx_i + b))^2$. Critical points are solutions of the two equations

$f_m = \sum_{i=1}^{n} -2x_i(y_i - (mx_i + b)) = 0$, so $\sum_{i=1}^{n}(-x_i y_i + mx_i^2 + bx_i) = 0$, or equivalently,

$$m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i;$$

and $f_b = \sum_{i=1}^{n} -2(y_i - (mx_i + b)) = 0$, so $\sum_{i=1}^{n}(-y_i + mx_i + b) = 0$, or equivalently,

$$m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i.$$

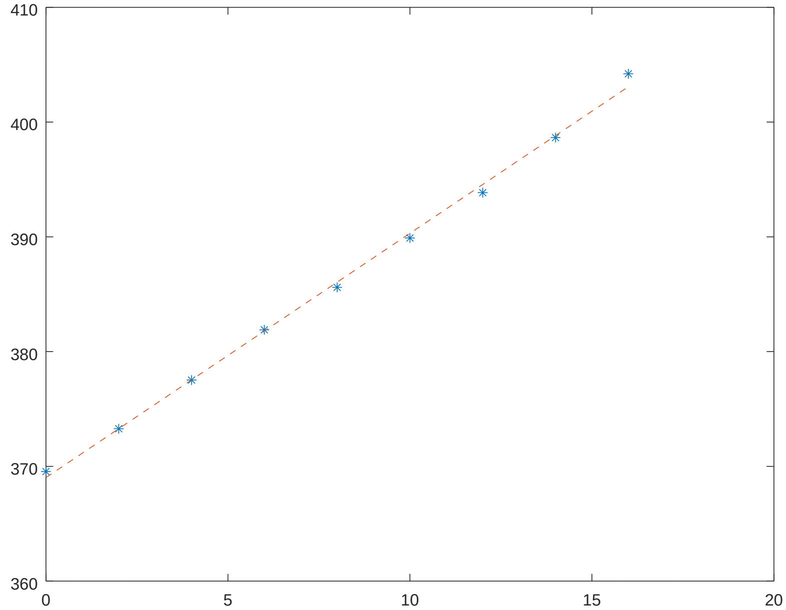
Thus the critical points are the solutions of the given equations. (Observe: given the data x_i and y_i, this is merely a 2×2 linear system!) We will not prove here that the critical point is a minimum.

b) We calculate $\sum x_i = 72$, $\sum x_i^2 = 816$, $\sum y_i = 3474.46$, $\sum x_i y_i = 28306.5$, while $n = 9$. So the equations are

$$\begin{cases} 72m + 9b = 3474.46 \\ 816m + 72b = 28306.5 \end{cases}$$

This gives $m \approx 2.128$ and $b \approx 369.02$.

c) For 2000 ($x = 0$), the predicted value (i.e., $mx+b$) is 369.02, vs. actual data 369.55. For 2008 ($x = 8$), the predicted value is 386.05, vs. actual data 385.60. For 2016 ($x = 16$), the predicted value is 403.07, vs. actual data 404.21. For 2100 ($x = 100$), the predicted value is 581.86 ppm. (A lot more than currently! Actual projections for 2100 range from 450 to 900 ppm depending on future carbon emissions.)



Problem 2. (pictures omitted)

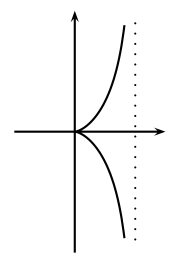
(a) $r = 3\sin\theta \Rightarrow r^2 = 3r\sin\theta \Leftrightarrow x^2 + y^2 = 3y$, i.e. $x^2 + (y - \frac{3}{2})^2 = (\frac{3}{2})^2$, circle of radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$.
(Note: multiplying both sides by r in the first step adds the extra solution $r = 0$, i.e. the origin; however the origin already lies on the circle, so this is of no consequence).

(b) $r = 5\sec\theta \Leftrightarrow r\cos\theta = 5 \Leftrightarrow x = 5$, vertical line through $(5, 0)$.

(c) $\theta = -\pi/3$: $y/x = \tan\theta = \tan(-\frac{\pi}{3}) = -\sqrt{3}$, i.e. $y = -\sqrt{3}x$, and $x \geq 0$: a half-line of slope $-\sqrt{3}$ through the origin (or the entire line if we allow $r < 0$).

(d) $r^2 \sin 2\theta = 2 \Leftrightarrow 2r^2 \sin\theta\cos\theta = 2 \Leftrightarrow 2xy = 2$, i.e. $y = 1/x$: hyperbola with asymptotes on the x and y axes.

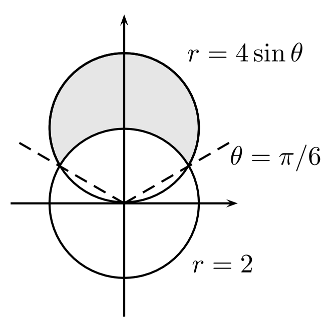
10.3 # 51: To show that $x = 1$ is an asymptote we must prove that $x \to 1$ as $r \to \infty$. $r \to \infty$ corresponds to $\sin\theta\tan\theta \to \infty$, i.e. $\theta \to \pm\pi/2$. (Note: $r(\theta + \pi) = -r(\theta)$, so if we allow all values of θ we trace the curve twice; instead we restrict ourselves to $-\pi/2 < \theta < \pi/2$, for which $r \geq 0$). Now, $x = r\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$, which does tend to 1 as $\theta \to \pm\pi/2$. (Note: $\theta \to (\pi/2)^-$ corresponds to $x \to 1$, $y \to +\infty$, while $\theta \to (-\pi/2)^+$ correspond to $x \to 1$, $y \to -\infty$).
Since $x = \sin^2\theta$ takes values ranging between 0 and 1, the curve is contained in the strip $0 \leq x \leq 1$ (and $x = 1$ is never reached). And, since $r(-\theta) = r(\theta)$, the curve is symmetric about the x-axis.



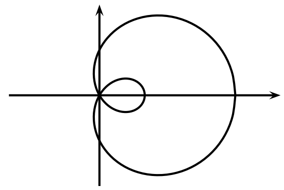
10.4 # 7: $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(4 + 3\sin\theta)^2\, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(16 + 24\sin\theta + 9\sin^2\theta)\, d\theta$.

Using parity, the portions of the integral from $-\pi/2$ to 0 and 0 to $\pi/2$ cancel out for $\sin\theta$, while they are equal for the other terms of the integrand; so
$A = \int_0^{\pi/2}(16 + 9\sin^2\theta)\, d\theta = \int_0^{\pi/2}(16 + \frac{9}{2}(1 - \cos 2\theta))\, d\theta = \left[\frac{41}{2}\theta - \frac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{41\pi}{4}$.

10.4 # 23: Inside $r = 4\sin\theta$, outside $r = 2$: the curves intersect when $4\sin\theta = 2$, i.e. $\sin\theta = \frac{1}{2}$, which gives $\theta = \pi/6$ or $5\pi/6$. See figure. To find the shaded area, we subtract the unshaded area (between dotted lines and $r = 2$) from the total area between the dotted lines and $r = 4\sin\theta$.
$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4\sin\theta)^2\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}2^2\, d\theta = \int_{\pi/6}^{5\pi/6}(8\sin^2\theta - 2)\, d\theta = \int_{\pi/6}^{5\pi/6}(2 - 4\cos 2\theta)\, d\theta$
$= [2\theta - 2\sin 2\theta]_{\pi/6}^{5\pi/6} = (\frac{5\pi}{3} + \sqrt{3}) - (\frac{\pi}{3} - \sqrt{3}) = \frac{4\pi}{3} + 2\sqrt{3}$.



10.4 # 35: $r = \frac{1}{2} + \cos\theta$ is positive for $\cos\theta > -\frac{1}{2}$, i.e. $-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$; this corresponds to the large loop; r is negative for $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, the small loop.



The area in the large loop is $2\int_0^{2\pi/3} \frac{1}{2}(\frac{1}{2}+\cos\theta)^2\,d\theta = \int_0^{2\pi/3}(\frac{1}{4}+\cos\theta+\cos^2\theta)\,d\theta = \int_0^{2\pi/3}(\frac{3}{4}+\cos\theta+\frac{1}{2}\cos 2\theta)\,d\theta = \left[\frac{3}{4}\theta+\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi/3} = (\frac{\pi}{2}+\frac{\sqrt{3}}{2}-\frac{1}{4}\frac{\sqrt{3}}{2}) - 0 = \frac{\pi}{2}+\frac{3\sqrt{3}}{8}$.

Similarly, the area in the small loop is
$2\int_{2\pi/3}^{\pi}\frac{1}{2}(\frac{1}{2}+\cos\theta)^2\,d\theta = \left[\frac{3}{4}\theta+\sin\theta+\frac{1}{4}\sin 2\theta\right]_{2\pi/3}^{\pi} = (\frac{3\pi}{4}) - (\frac{\pi}{2}+\frac{3\sqrt{3}}{8}) = \frac{\pi}{4}-\frac{3\sqrt{3}}{8}$.

Subtracting, the desired area is $(\frac{\pi}{2}+\frac{3\sqrt{3}}{8}) - (\frac{\pi}{4}-\frac{3\sqrt{3}}{8}) = \frac{\pi}{4}+\frac{3\sqrt{3}}{4}$.

15.1 # 24: $\displaystyle\int_0^1 xy\sqrt{x^2+y^2}\,dy = \left[\frac{1}{3}x(x^2+y^2)^{3/2}\right]_0^1 = \frac{1}{3}x(x^2+1)^{3/2} - \frac{1}{3}x^4.$

So $\displaystyle\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx = \frac{1}{3}\int_0^1 x(x^2+1)^{3/2} - x^4\,dx =$

$$= \frac{1}{15}\left[(x^2+1)^{5/2} - x^5\right]_0^1 = \frac{1}{15}\left[(2^{5/2}-1)-(1-0)\right] = \frac{4\sqrt{2}-2}{15}.$$

15.1 # 29: $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx.$

Inner: $\displaystyle\left[\frac{x}{x^2+1}\,\frac{y^3}{3}\right]_{-3}^3 = \frac{x}{x^2+1}\left(\frac{27-(-27)}{3}\right) = 18\frac{x}{x^2+1}.$

Outer: $\displaystyle\int_0^1 18\frac{x}{x^2+1}\,dx = 18\left[\frac{1}{2}\ln(x^2+1)\right]_0^1 = 9\ln 2.$

15.1 # 32: $\displaystyle\iint_R \frac{x}{1+xy}\,dA = \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \Big[\ln(1+xy)\Big]_{y=0}^{y=1}\,dx$

$$= \int_0^1 \ln(1+x)\,dx = [(1+x)\ln(1+x) - x]_0^1 = 2\ln 2 - 1.$$

(Note: integrating in the other order is slightly harder.)

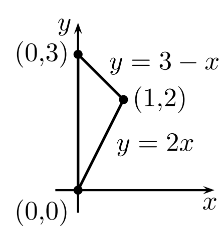
Problem 3: a) $\int_0^1\int_{x^2}^{x}(1+2y)\,dy\,dx = \int_0^1 [y+y^2]_{y=x^2}^{y=x}\,dx = \int_0^1((x+x^2)-(x^2+x^4))\,dx$

$$= \int_0^1 (x-x^4)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{2}-\frac{1}{5} = \frac{3}{10}.$$

b) $\displaystyle\iint_D \frac{y}{x^5+1}\,dA = \int_0^1\int_0^{x^2}\frac{y}{x^5+1}\,dy\,dx = \int_0^1\left[\frac{y^2/2}{x^5+1}\right]_0^{x^2}dx = \int_0^1 \frac{1}{2}\frac{x^4}{x^5+1}\,dx$

$$= \frac{1}{10}\ln(x^5+1)\Big]_0^1 = \frac{1}{10}\ln 2.$$

c) Setting up the bounds (see figure): $\displaystyle\int_0^1\int_{2x}^{3-x} 2xy\,dy\,dx$



(or also: $\int_0^2\int_0^{y/2} 2xy\,dx\,dy + \int_2^3\int_0^{3-y} 2xy\,dx\,dy$)

Inner: $\int_{2x}^{3-x} 2xy\,dy = \left[xy^2\right]_{y=2x}^{y=3-x} = x(3-x)^2 - 4x^3 = 9x - 6x^2 - 3x^3.$

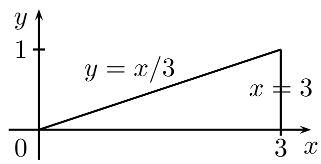
Outer: $\int_0^1 9x - 6x^2 - 3x^3\,dx = \left[\frac{9}{2}x^2 - 2x^3 - \frac{3}{4}x^4\right]_0^1 = \frac{7}{4}.$

15.2 # 27: The plane $2x + y + z = 4$ (or $z = 4 - 2x - y$) intersects the xy-plane ($z = 0$) along the line $2x+y = 4$ (with x-intercept 2 and y-intercept 4), or $y = 4-2x$. So the region of integration is the triangle with vertices (0,0), (2,0) and (0,4): $x \geq 0$, $0 \leq y \leq 4 - 2x$, and the integrand is $4 - 2x - y$.

$$\int_0^2 \int_0^{4-2x} (4-2x-y)\,dy\,dx = \int_0^2 \left[(4-2x)y - \frac{1}{2}y^2\right]_0^{4-2x} dx = \int_0^2 (4-2x)^2 - \frac{1}{2}(4-2x)^2\,dx =$$

$$= \int_0^2 \tfrac{1}{2}(4-2x)^2\,dx = -\tfrac{1}{12}(4-2x)^3\Big|_0^2 = \tfrac{4^3}{12} = \tfrac{16}{3}.$$

15.2 # 51:



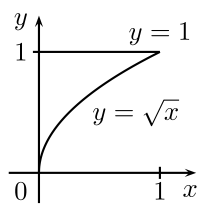
Region of integration: $3y \leq x \leq 3$, $0 \leq y \leq 1$

or equivalently: $0 \leq y \leq x/3$, $0 \leq x \leq 3$

So $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx.$

Inner: $\displaystyle\left[ye^{x^2}\right]_0^{x/3} = \frac{1}{3}xe^{x^2}$. Outer: $\displaystyle\int_0^3 \frac{1}{3}xe^{x^2}\,dx = \left[\frac{1}{6}e^{x^2}\right]_0^3 = \frac{1}{6}(e^9 - 1).$

15.2 # 53:



Region of integration: $\sqrt{x} \leq y \leq 1$, $0 \leq x \leq 1$

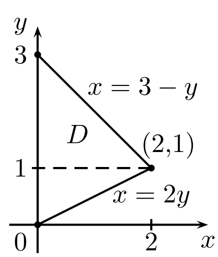
or equivalently: $0 \leq x \leq y^2$, $0 \leq y \leq 1$

So $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx = \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy.$

Inner: $\displaystyle\left[x\sqrt{y^3+1}\right]_0^{y^2} = y^2\sqrt{y^3+1}.$

Outer: $\displaystyle\int_0^1 y^2\sqrt{y^3+1}\,dy = \tfrac{2}{9}(y^3+1)^{3/2}\Big|_0^1 = \frac{2}{9}(2\sqrt{2}-1).$

15.2 # 64:



$$\int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy$$

$$= \iint_D f(x,y)\,dA = \int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx.$$