Math 53 Homework 6 – Solutions

14.7 # 36: $f_x = y^2$ and $f_y = 2xy$, and the critical points correspond to $y = 0$; so there are none in the interior of D (the portion of the disk of radius $\sqrt{3}$ lying in the first quadrant). So the maximum and minimum occur at the boundaries. On the x and y axes, $f(x, 0) = f(0, y) = 0$. On the circular edge, $y = \sqrt{3 - x^2}$ so we consider $g(x) = f(x, \sqrt{3 - x^2}) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0$ $\Leftrightarrow x = 1$. We get that the maximum value on this edge is $g(1) = f(1, \sqrt{2}) = 2$, and the minimum value is 0, occurring both at $x = 0$ and $x = \sqrt{3}$. Thus the absolute maximum of f on D is $f(1, \sqrt{2}) = 2$, and the absolute minimum is 0 which occurs at all points along the x and y axes.

Problem 1: $f(x, y) = x^4 + y^4 - 4xy + 2 \Rightarrow f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$. Then $f_x = 0 \Leftrightarrow y = x^3$, $f_y = 0 \Leftrightarrow x = y^3$; substituting, we get $x = x^9$, or $x(x^8 - 1) = 0$, which gives the three solutions $x = 0$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$. Of those, only $(1, 1)$ is inside the given region D, and $f(1, 1) = 0$. (The second derivative test would tell us that $(1, 1)$ is a local minimum, since $f_{xx} = 12x^2 = 12$, $f_{xy} = -4$, $f_{yy} = 12y^2 = 12$, and $12 \cdot 12 - (-4)^2 > 0$.)

For $y = 0$, $0 \le x \le 3$ (bottom edge): $f(x, 0) = x^4 + 2$, attaining its minimum at $x = 0$ ($f(0, 0) = 2$) and its maximum at $x = 3$ ($f(3, 0) = 83$).

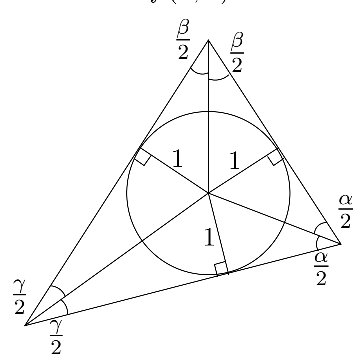
For $x = 3$, $0 \le y \le 2$ (right edge): $f(3, y) = y^4 - 12y + 83$, a polynomial which attains its minimum for $\frac{d}{dy}(y^4 - 12y + 83) = 4y^3 - 12 = 0$ or $y = \sqrt[3]{3}$, with $f(3, \sqrt[3]{3}) = -9\sqrt[3]{3} + 83 \approx 70$, and its maximum at $y = 0$, $f(3, 0) = 83$.

For $y = 2$, $0 \le x \le 3$ (top edge): $f(x, 2) = x^4 - 8x + 18$, attaining its minimum for $\frac{d}{dx}(x^4 - 8x + 18) = 4x^3 - 8x = 0$ or $x = \sqrt[3]{2}$, with $f(\sqrt[3]{2}, 2) = -6\sqrt[3]{2} + 18 \approx 10.4$, and its maximum at $x = 3$, $f(3, 2) = 75$ ($> f(0, 2) = 18$).

For $x = 0$, $0 \le y \le 2$ (left edge): $f(0, y) = y^4 + 2$, attaining its minimum at $x = 0$ ($f(0, 0) = 2$) and its maximum at $y = 2$ ($f(0, 2) = 18$).

Comparing the various possible minima and maxima we have found, the absolute min imum of f on D is $f(1, 1) = 0$, and the absolute maximum is $f(3, 0) = 83$.

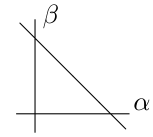
Problem 2. a) The triangle splits into 6 right triangles, two with area $\frac{1}{2} \cot \frac{\alpha}{2}$, two with area $\frac{1}{2} \cot \frac{\beta}{2}$, and the last two with area $\frac{1}{2} \cot \frac{\gamma}{2}$. Hence



$$A = \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2}.$$

$\alpha + \beta + \gamma = \pi \Rightarrow \cot \frac{\gamma}{2} = \cot(\frac{\pi}{2} - \frac{\alpha + \beta}{2}) = \tan(\frac{\alpha + \beta}{2})$.
So $A = \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \tan \frac{\alpha + \beta}{2}$.

b) The set of possible values for the angles is given by $\alpha > 0$, $\beta > 0$, $\alpha + \beta < \pi$. This is a triangular region in the (α, β) coordinates, with boundaries $\alpha = 0$, $\beta = 0$, and $\alpha + \beta = \pi$.



$$\frac{\partial A}{\partial \alpha} = -\frac{1}{2} \csc^2 \frac{\alpha}{2} + \frac{1}{2} \sec^2 \frac{\alpha + \beta}{2} \quad \text{and} \quad \frac{\partial A}{\partial \beta} = -\frac{1}{2} \csc^2 \frac{\beta}{2} + \frac{1}{2} \sec^2 \frac{\alpha + \beta}{2}.$$

So the critical points are solutions of $-\csc^2\frac{\alpha}{2} = \csc^2\frac{\beta}{2} = \sec^2\frac{\alpha+\beta}{2}$, or equivalently $\sin^2\frac{\alpha}{2} = \sin^2\frac{\beta}{2} = \cos^2\frac{\alpha+\beta}{2}$. Since these angles are between 0 and $\pi/2$, this implies $\alpha/2 = \beta/2 = \pi/2 - (\alpha+\beta)/2$. Hence the only critical point is $\alpha = \beta = \pi/3$.

c) At the critical point $\alpha = \beta = \frac{\pi}{3}$ we have $A = 3\sqrt{3}$. The boundary of the (triangular) domain where A is defined consists of three parts: near the boundary $\alpha = 0$, the term $\cot\frac{\alpha}{2}$ increases to infinity. Similarly, $\cot\frac{\beta}{2}$ increases to infinity when β approaches 0. Finally, near $\alpha + \beta = \pi$ the last term $\tan\frac{\alpha+\beta}{2}$ increases to infinity. So, no matter which boundary of the domain we consider, the value of A increases to infinity as we approach it.

By comparing the values of A at the critical point and near the boundary, the minimum of A is $3\sqrt{3}$, reached for $\alpha = \beta = \gamma = \frac{\pi}{3}$, corresponding to an equilateral triangle. On the other hand, the value of A can become arbitrarily large when one of α, β or γ approaches zero, corresponding to the degenerate case of a very long and thin triangle, with two sides almost parallel to each other and intersecting at a vertex that lies very far from the incenter.

14.8 # 3: $f(x,y) = x^2 - y^2$, $g(x,y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2x, -2y\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. Hence $\nabla f = \lambda\nabla g$ becomes: $2x = 2x\lambda$, $-2y = 2y\lambda$.

If $x \neq 0$ then $\lambda = 1$ and the second equation gives $y = 0$. The constraint $x^2+y^2 = 1$ implies that $x = \pm 1$. This gives the two points $(\pm 1, 0)$.

Whereas if $x = 0$ then $x^2 + y^2 = 1$ gives $y = \pm 1$, and the multiplier equations hold with $\lambda = -1$. This gives $(0, \pm 1)$.

Checking the values of f at these points, $(\pm 1, 0)$ give the maxima, with $f = 1$, while $(0, \pm 1)$ give the minima, with $f = -1$.

14.8 # 9: $f(x,y,z) = xy^2z$, $g(x,y,z) = x^2 + y^2 + z^2 = 4 \Rightarrow \nabla f = \langle y^2z, 2xyz, xy^2\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$. Thus $\nabla f = \lambda\nabla g$ becomes:

$$y^2z = 2\lambda x, \quad 2xyz = 2\lambda y, \quad xy^2 = 2\lambda z. \tag{1}$$

Multiplying the first equation by x, the second by $y/2$, the third by z, we get $xy^2z = 2\lambda x^2 = \lambda y^2 = 2\lambda z^2$. There are two cases.

If $\lambda = 0$, then (1) becomes $y^2z = 0$, $xyz = 0$, $xy^2 = 0$, so either $y = 0$ or $x = z = 0$. This gives the points $(0, \pm 2, 0)$ and the circle $\{(x, 0, z) \mid x^2 + z^2 = 4\}$ in the xz-plane. Either way, $f = 0$ at all these points.

Otherwise, $2\lambda x^2 = \lambda y^2 = 2\lambda z^2$ implies $2x^2 = y^2 = 2z^2$, and the constraint $x^2 + y^2 + z^2 = 4$ becomes $4x^2 = 4$, so $x^2 = 1$, $y^2 = 2$, $z^2 = 1$. This gives the 8 points $(\pm 1, \pm\sqrt{2}, \pm 1)$, at which $f = \pm 2$. Hence the maximum value of f on the given sphere is 2, occurring at the four points $(1, \pm\sqrt{2}, 1)$ and $(-1, \pm\sqrt{2}, -1)$; and the minimum value is -2, occurring at the four other points.

14.8 # 33: we want to minimize $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$ (the squared distance) subject to the constraint $g(x,y,z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda\nabla g$ becomes

$$2(x-4) = 2\lambda x, \quad 2(y-2) = 2\lambda y, \quad 2z = -2\lambda z.$$

Considering the last equation, either $z = 0$ (in which case the equation of the cone gives $x = y = 0$, which do not satisfy the other multiplier equations), or

$\lambda = -1$. Plugging $\lambda = -1$ into the first two equations, we get $2(x-4) = -2x$ and $2(y-2) = -2y$. So $x = 2$, $y = 1$, and using the equation of the cone, $z = \pm\sqrt{5}$.

14.8 # 39: Let $f(x, y, z) = xyz$ (volume of the box), $g(x, y, z) = x + 2y + 3z = 6$ (the constraint). Then $\nabla f = \langle yz, xz, xy\rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda\rangle$. Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies that either $\lambda = 0$ and two of x, y, z are zero (but then $f = 0$, not a maximum); or $y = \frac{1}{2}x$ and $z = \frac{1}{3}x$. Plugging into the constraint equation, we get $3x = 6$, so $x = 2$, $y = 1$, and $z = \frac{2}{3}$. The largest volume is $f(2, 1, \frac{2}{3}) = \frac{4}{3}$.

Problem 3. a) Minimizing $f(x, y, z) = (x-4)^2 + (y-2)^2 + (z-3)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 - 6z = 0$ gives the Lagrange multiplier equations $\nabla f = \lambda \nabla g$, or

$$\begin{aligned} 2(x-4) &= \lambda(2x) \\ 2(y-2) &= \lambda(2y) \\ 2(z-3) &= \lambda(-6) \end{aligned}$$

The system now has four variables, so we also need to remember a fourth equation, the constraint equation, $x^2 + y^2 - 6z = 0$.

b)
$$\begin{aligned} x - 4 = \lambda x &\iff (1-\lambda)x = 4 \iff x = 4/(1-\lambda) \\ y - 2 = \lambda y &\iff (1-\lambda)y = 2 \iff y = 2/(1-\lambda) \\ z - 3 = -3\lambda &\iff z = 3(1-\lambda) \end{aligned}$$

Thus $x = \dfrac{12}{z}$, $y = \dfrac{6}{z}$, and the constraint equation becomes $\left(\dfrac{12}{z}\right)^2 + \left(\dfrac{6}{z}\right)^2 - 6z = 0$. So $180/z^2 = 6z$, or $z^3 = 30$: so $z = \sqrt[3]{30} \approx 3.10723$. Using $x = 12/z$ and $y = 6/z$ one finds

$$(x, y, z) = \left(\tfrac{12}{\sqrt[3]{30}}, \tfrac{6}{\sqrt[3]{30}}, \sqrt[3]{30}\right) \approx (3.86196, 1.93098, 3.10723).$$

The answer found in HW 6 was $(4 - \frac{4}{29}, 2 - \frac{2}{29}, 3 + \frac{3}{29}) \approx (3.86207, 1.93103, 3.10345)$ which is within $1/100$ of the exact solution (the largest difference is $\Delta z \approx 0.00378$).

Problem 4. a) $w = x^2 + y^2 + z^2$ where $y^2 + xz = 2$:

(i) $f(x, y, z) = x^2 + y^2 + z^2$, and $\partial f/\partial x = 2x$. In this case y and z are being held constant (but $y^2 + xz$ is not being held constant).

(ii) we use the relation $y^2 + xz = 2$ to solve for z: namely, $z = (2 - y^2)/x$. Therefore $w = g(x, y) = x^2 + y^2 + \dfrac{(2-y^2)^2}{x^2}$. This gives $\left(\dfrac{\partial w}{\partial x}\right)_y = \dfrac{\partial g}{\partial x} = 2x - \dfrac{2(2-y^2)^2}{x^3}$.

Here we differentiate with respect to x while y is being held constant, but z varies (so that the relation $y^2 + xz = 2$ remains valid).

(iii) we use the relation $y^2 + xz = 2$ to eliminate y: namely, $y^2 = 2 - xz$, so $w = h(x, z) = x^2 + (2 - xz) + z^2$. This gives $\left(\dfrac{\partial w}{\partial x}\right)_z = \dfrac{\partial h}{\partial x} = 2x - z$.

Here we differentiate with respect to x while z is being held constant, but y varies (so that the relation $y^2 + xz = 2$ remains valid).

(To see that these partial derivatives are really different, observe that for $(x, y, z) = (1, 1, 1)$, we get $\partial f/\partial x = 2$, but $\partial g/\partial x = 0$ and $\partial h/\partial x = 1$! This is of course because these quantities correspond to scenarios where different quantities are being held constant.)

b) Differentiating $w = x^2+y^2+z^2$, we get: $dw = 2x\,dx+2y\,dy+2z\,dz$. Differentiating the constraint equation $y^2 + xz = 2$, we get: $z\,dx + 2y\,dy + x\,dz = 0$.

Treating x, y as independent variables: $z\,dx+2y\,dy+x\,dz = 0 \Rightarrow dz = -\frac{z}{x}\,dx - \frac{2y}{x}\,dy$. Substituting into dw, we get:

$$dw = 2x\,dx + 2y\,dy + 2z\left(-\frac{z}{x}\,dx - \frac{2y}{x}\,dy\right) = \left(2x - \frac{2z^2}{x}\right)dx + \left(2y - \frac{4yz}{x}\right)dy.$$

So $\left(\frac{\partial w}{\partial x}\right)_y = 2x - \frac{2z^2}{x}$ (which is consistent with the formula in (a)).

Next, viewing x, z as independent: $z\,dx + 2y\,dy + x\,dz = 0 \Rightarrow dy = -\frac{z}{2y}\,dx - \frac{x}{2y}\,dz$. Substituting into dw:

$$dw = 2x\,dx + 2y\left(-\frac{z}{2y}\,dx - \frac{x}{2y}\,dz\right) + 2z\,dz = (2x - z)\,dx + (2z - x)\,dz.$$

So $\left(\frac{\partial w}{\partial x}\right)_z = 2x - z$.