

Math 53 Homework 6

Due Wednesday 10/4/17 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 9/25 – Max-min problems continued

• **Read:** section 14.7.

• **Work:** 14.7: (31), 36.¹

Problems 1 and 2 below.

Wednesday 9/27 – Non-independent variables, Lagrange multipliers

• **Read:** section 14.8.

• **Work:** 14.8: (1), 3², (7), 9³, (29), 33, (37), 39.⁴

Problems 3 and 4 below.

Friday 9/29 – Midterm 1 Review

Problem 1.

Find the absolute maximum and minimum values of $f(x, y) = x^4 + y^4 - 4xy + 2$ in the region $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Problem 2.

Consider a triangle in the plane, with angles α, β, γ . Assume that the radius of its incircle is equal to 1.

a) By decomposing the triangle into six right triangles having the incenter as a common vertex, express the area A of the triangle in terms of α, β, γ (your answer should be a symmetric expression). Then use your result to show that A can be expressed as a function of the two variables α and β by the formula

$$A = \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \tan \frac{\alpha + \beta}{2}.$$

b) What is the set of possible values for α and β ? Find all the critical points of the function A in this region.

c) By computing the values of A at the critical points and its behavior on the boundary of the region where it is defined, find the maximum and the minimum of A (justify your answer). Describe the shapes of the triangles corresponding to these two situations.

¹**7th ed:** 14.7: (29), 34.

²**7th ed:** do the 8th ed version: $f(x, y) = x^2 - y^2, x^2 + y^2 = 1$.

³**7th ed:** do the 8th ed version: $f(x, y, z) = xy^2z, x^2 + y^2 + z^2 = 4$.

⁴**7th ed:** (1), 3^{*}, (7), 9^{*}, (27), 31, (35), 37.

Problem 3. This problem uses Lagrangian multipliers to find an exact answer to Problem 3(b) of HW 5.

a) Use the method of Lagrange multipliers to write down the system of equations satisfied by the point closest to $(4, 2, 3)$ at which $x^2 + y^2 - 6z = 0$. (Hint: it is easier to minimize the square of the distance).

b) Solve the equations you found in (a) to get the exact location of the point. Then use a calculator to evaluate your answers to five decimal places. Compare your answers with the approximate solution you found in HW 5 Problem 3(b). Was each coordinate of the approximate answer within $1/100$ of the exact answer?

Problem 4. – Non-independent variables.

The goal of this problem is to illustrate a subtlety in the definition of partial derivatives when variables are not independent. This is an important issue in thermodynamics and some other fields; here we consider just a simple mathematical example.

Let $w = x^2 + y^2 + z^2$, where the variables x, y, z are related to each other by the equation $y^2 + xz = 2$. We can give three different meanings to the quantity $\partial w / \partial x$.

(i) We can treat the variables x, y, z as independent, and write $w = f(x, y, z) = x^2 + y^2 + z^2$. Then we consider $\partial f / \partial x$.

(ii) We can treat x and y as independent variables, with z implicitly defined as a function of x and y by the relation $y^2 + xz = 2$. Then w is given by some function $g(x, y)$, and we consider $\frac{\partial g}{\partial x}$. This quantity is sometimes denoted by $\left(\frac{\partial w}{\partial x}\right)_y$.

(iii) We can treat x and z as independent variables, with y implicitly defined as a function of x and z by the relation $y^2 + xz = 2$. Then w is given by some function $h(x, z)$, and we consider $\frac{\partial h}{\partial x}$. This quantity is sometimes denoted by $\left(\frac{\partial w}{\partial x}\right)_z$.

a) Determine the functions f, g, h , and calculate $\partial f / \partial x$, $\partial g / \partial x$, and $\partial h / \partial x$. Also say in each case which quantities are being held constant and which ones are not.

(Optional: compare the values of these partial derivatives at $(x, y, z) = (1, 1, 1)$ to convince yourself that they are really different.)

b) Now we try a more systematic approach, which would work even if we were unable to find expressions for the functions $g(x, y)$ and $h(x, z)$ by solving the constraint equation.

First, express the differential dw in terms of dx , dy and dz , and also differentiate the constraint equation to find a relation between dx , dy and dz . Then, use this relation to eliminate dz and express dw in terms of dx and dy ; use this to find $\left(\frac{\partial w}{\partial x}\right)_y$. Similarly, eliminate dy to find $\left(\frac{\partial w}{\partial x}\right)_z$.

(Your answers might be different from those in part (a), but are they consistent with them?)