

Math 53 Homework 5 – Solutions

14.5 # 5:
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 2te^{y/z} - \frac{x}{z}e^{y/z} - 2\frac{xy}{z^2}e^{y/z}$$

$$= \left(2t - \frac{t^2}{1+2t} - \frac{2t^2(1-t)}{(1+2t)^2}\right)e^{(1-t)/(1+2t)}.$$

14.5 # 7:
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 5(x-y)^4(2st) - 5(x-y)^4t^2 = 5(x-y)^4(2st - t^2).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 5(x-y)^4s^2 - 5(x-y)^4(2st) = 5(x-y)^4(s^2 - 2st).$$

14.5 # 15: $g(u, v) = f(x(u, v), y(u, v))$ where $x(u, v) = e^u + \sin v$ and $y(u, v) = e^u + \cos v$. So $g_u = f_x x_u + f_y y_u = e^u f_x + e^u f_y$, and $g_v = f_x x_v + f_y y_v = \cos v f_x - \sin v f_y$. At $(u, v) = (0, 0)$, $(x, y) = (1, 2)$; so $g_u(0, 0) = 1 \cdot f_x(1, 2) + 1 \cdot f_y(1, 2) = 2 + 5 = 7$, and $g_v(0, 0) = 1 \cdot f_x(1, 2) - 0 \cdot f_y(1, 2) = 2$.

14.5 # 18:
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u};$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}.$$

14.5 # 43: Call the sides a and b , and the angle between them θ ; so the area $A = \frac{1}{2}ab \sin \theta$ is assumed to be constant. So $\frac{dA}{dt} = \frac{\partial A}{\partial a} \frac{da}{dt} + \frac{\partial A}{\partial b} \frac{db}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} = 0$, i.e., $\frac{1}{2}b \sin \theta \frac{da}{dt} + \frac{1}{2}a \sin \theta \frac{db}{dt} + \frac{1}{2}ab \cos \theta \frac{d\theta}{dt} = 0$. Solving for $d\theta/dt$, we get:

$$\frac{d\theta}{dt} = -\frac{b \sin \theta \frac{da}{dt} + a \sin \theta \frac{db}{dt}}{ab \cos \theta}.$$

Using the given values for $a, b, \theta, da/dt, db/dt$, we get

$$\frac{d\theta}{dt} = -\frac{30 \cdot \frac{1}{2} \cdot 3 + 20 \cdot \frac{1}{2} \cdot (-2)}{20 \cdot 30 \cdot \frac{\sqrt{3}}{2}} = -\frac{5}{60\sqrt{3}} \approx -0.048 \text{ rad/s}.$$

14.5 # 45: (a) chain rule: $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r \sin \theta) + \frac{\partial z}{\partial y} r \cos \theta$.

(b) $\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + 2\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta$, and

$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2 \theta - 2\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} r^2 \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2 \theta$. So

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] (\cos^2 \theta + \sin^2 \theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

14.5 # 51: $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2s \frac{\partial z}{\partial x} + 2r \frac{\partial z}{\partial y}$. So

$$\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial}{\partial r} \left(2s \frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial r} \left(2r \frac{\partial z}{\partial y}\right) = 2s \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x}\right) + 2r \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y}\right) + 2 \frac{\partial z}{\partial y}$$

$$= 2s \left(\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x}\right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x}\right) \frac{\partial y}{\partial r}\right) + 2r \left(\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y}\right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y}\right) \frac{\partial y}{\partial r}\right) + 2 \frac{\partial z}{\partial y}$$

$$\begin{aligned}
&= 4rs \frac{\partial^2 z}{\partial x^2} + 4s^2 \frac{\partial^2 z}{\partial y \partial x} + 4r^2 \frac{\partial^2 z}{\partial x \partial y} + 4rs \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial y} \\
&= 4rs \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) + 4(r^2 + s^2) \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial z}{\partial y}.
\end{aligned}$$

Explanation: to compute $\partial^2 z / \partial r \partial s$, we differentiate the expression for $\partial z / \partial s$ with respect to r (keeping s constant). The first step involves the product rule. The second one is more subtle. To calculate $\frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x} \right)$, we use the chain rule once more. If this feels confusing, just set $g = \partial z / \partial x$ (think of this as some new function of x and y), and note that we are trying to calculate $\frac{\partial g}{\partial r}$. The chain rule gives $\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial r}$. Remembering that $g = \partial z / \partial x$, the first partials of g with respect to x and y are actually second partial derivatives of z . The chain rule is used similarly to differentiate $\partial z / \partial y$ with respect to r .

14.6 # 1: We can approximate the directional derivative at K by the average rate of change of pressure between the points where the line through K and S (red on the figure) intersects the contour lines closest to K. In this case we measure that, going about 1/6 of the way towards S, ($\Delta s \approx 50$ km), the pressure drops from 1000 to 996 mb ($\Delta P \approx -4$ mb). So $D_{\hat{u}}P \approx \frac{\Delta P}{\Delta s} \approx \frac{-4}{50} = -0.08$ (millibars per km).

14.6 # 9: (a) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \rangle$.

(b) at $(x, y, z) = (2, -1, 1)$, $\nabla f = \langle -3, 2, 2 \rangle$.

(c) $\hat{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$ is a unit vector, so $D_{\hat{u}}f = \nabla f \cdot \hat{u} = \langle -3, 2, 2 \rangle \cdot \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle = \frac{2}{5}$.

14.6 # 27: (a) Given a unit direction vector \hat{u} , recall that $D_{\hat{u}}f = \nabla f \cdot \hat{u} = |\nabla f| \cos \theta$ (where θ is the angle between ∇f and \hat{u}). Since the minimum value of $\cos \theta$ is -1 , occurring for $\theta = \pi$, the minimum value of $D_{\hat{u}}f$ is $-|\nabla f|$ and occurs when \hat{u} is in the opposite direction of ∇f .

(b) $\nabla f = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 \rangle$, so at the point $(2, -3)$ f decreases fastest in the direction of $-\nabla f(2, -3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$ (or the corresponding unit vector).

14.6 # 38: $\nabla f(4, 6)$ is perpendicular to the level curve of f that passes through $(4, 6)$; so we sketch a portion of level curve through $(4, 6)$ (using the nearby level curves as guidelines), and draw a line perpendicular to it. The direction of the gradient vector is parallel to this line, and pointing towards increasing values of f . (towards the lower-right, making about a 65° angle with the horizontal direction).

Next we estimate the magnitude $|\nabla f|$, which equals the directional derivative of f at $(4, 6)$ in the direction of ∇f . We estimate this by finding the average rate of change along the direction perpendicular to the level curve. The points where the line previously drawn intersects the contour lines $f = -2$ and $f = -3$ are ≈ 0.5 units apart, so $\Delta f = 1$ and $\Delta s \approx 0.5$, giving $|\nabla f| \approx \frac{\Delta f}{\Delta s} \approx \frac{1}{0.5} = 2$. Hence we sketch the gradient vector with length 2. (Diagram omitted).

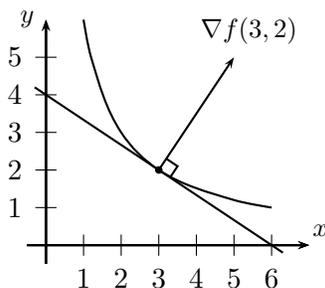
(Note: we could also have tried to estimate f_x and f_y separately and use those to sketch ∇f ; this is much less accurate, especially concerning the direction of ∇f .)

14.6 # 41: Let $f(x, y, z) = 2(x - 2)^2 + (y - 1)^2 + (z - 3)^2$: then we are considering the level surface $f = 10$. $\nabla f = \langle 4(x - 2), 2(y - 1), 2(z - 3) \rangle$, so $\nabla f(3, 3, 5) = \langle 4, 4, 4 \rangle$.

(a) $\nabla f(3, 5, 5) = \langle 4, 4, 4 \rangle$ is a normal vector for the tangent plane at $(3, 3, 5)$, so an equation of the tangent plane is $4(x-3) + 4(y-3) + 4(z-5) = 0$, or $4x + 4y + 4z = 44$ (or $x + y + z = 11$).

(b) The normal line has direction $\langle 4, 4, 4 \rangle$, so parametric equations are $x = 3 + 4t$, $y = 3 + 4t$, $z = 5 + 4t$. (or using $\langle 1, 1, 1 \rangle$, $x = 3 + t$, $y = 3 + t$, $z = 5 + t$).

14.6 # 49: $\nabla f = \langle y, x \rangle$, so $\nabla f(3, 2) = \langle 2, 3 \rangle$. So the tangent line has equation $\langle 2, 3 \rangle \cdot \langle x - 3, y - 2 \rangle = 0 \Rightarrow 2(x - 3) + 3(y - 2) = 0$, which simplifies to $2x + 3y = 12$.



14.6 # 56: first note that the point $(1, 1, 2)$ is on both surfaces. Let $f(x, y, z) = 3x^2 + 2y^2 + z^2$, so the ellipsoid is $f = 9$: then $\nabla f = \langle 6x, 4y, 2z \rangle$, so the tangent plane to the ellipsoid has normal vector $\nabla f(1, 1, 2) = \langle 6, 4, 4 \rangle$, and an equation of the tangent plane is $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is the level surface $g = 0$ where $g(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$, and $\nabla g = \langle 2x - 8, 2y - 6, 2z - 8 \rangle$. So the tangent plane at $(1, 1, 2)$ has normal vector $\nabla g(1, 1, 2) = \langle -6, -4, -4 \rangle$, giving the equation $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. The tangent planes are the same, so the surfaces are tangent to each other at $(1, 1, 2)$.

(Note: it would have been enough to show that the normal vectors are parallel to each other, without determining the equations of the tangent planes.)

Problem 1. a) (i) We need the vector $\hat{u} = \langle a, b \rangle$ to be tangent to the level curve through $(\frac{3}{2}, \frac{1}{2})$ (whose shape we can estimate from the neighboring ones). Indeed, the directional derivative $D_{\hat{u}}f = \nabla f \cdot \hat{u}$ is zero when \hat{u} is perpendicular to ∇f , i.e. tangent to the level curve through $(\frac{3}{2}, \frac{1}{2})$.

Hence, the two directions in which $df/dt = 0$ are the two unit vectors which are tangent to the level curve at $(\frac{3}{2}, \frac{1}{2})$. (One is about 30° counterclockwise from \hat{i} , the other is in the opposite direction i.e. about 150° clockwise from \hat{i}).

Estimating the shape of the level curve through $(\frac{3}{2}, \frac{1}{2})$ and drawing its tangent line L , we find that L lies on the side of the level curve where $f(x, y) \geq f(\frac{3}{2}, \frac{1}{2})$. So, moving along L in either direction, the value of f reaches a minimum at $(\frac{3}{2}, \frac{1}{2})$.

(ii) The directional derivative is largest in the direction of the gradient ∇f (perpendicular to level curve, towards larger values of f), i.e. about 120° counterclockwise from \hat{i} . It is smallest in the direction of $-\nabla f$, i.e. about 60° clockwise from \hat{i} . The nearest given level curves ($f = -0.8$ and $f = -1$) are about 0.5 cm apart, i.e. 0.1 unit, in that direction, hence we estimate $df/dt \simeq \Delta f / \Delta t \simeq 0.2 / 0.1 = 2$ in the direction of ∇f , and -2 in the opposite direction.

b) $f_x = 3x^2 - 6x + y + 1$, and $f_y = x - 2y + 1$, so $f_x(\frac{3}{2}, \frac{1}{2}) = -\frac{3}{4}$, $f_y(\frac{3}{2}, \frac{1}{2}) = \frac{3}{2}$. Using the chain rule (or the definition of the directional derivative), $df/dt = f_x dx/dt + f_y dy/dt = -\frac{3}{4}a + \frac{3}{2}b$ when $t = 0$.

(i) We want $-\frac{3}{4}a + \frac{3}{2}b = 0$, i.e. $a = 2b$. So we want unit vectors of the form $\langle 2b, b \rangle$. We need $|\langle 2b, b \rangle| = \sqrt{5b^2} = 1$, hence $b = \pm \frac{1}{\sqrt{5}}$. The two solutions are therefore $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$ and $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$.

(ii) The largest directional derivative is in the direction of $\nabla f = \langle -\frac{3}{4}, \frac{3}{2} \rangle$, i.e. the unit vector $\langle a, b \rangle = \frac{\nabla f}{|\nabla f|} = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$. The smallest value is in the direction of $-\nabla f$, i.e. the unit vector $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$. The directional derivatives in those two directions are $\nabla f \cdot \langle a, b \rangle = \pm |\nabla f| = \pm \frac{3}{4}\sqrt{5} \approx \pm 1.68$.

Problem 2: $\nabla f(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$.

$\nabla f(3, 6, -2) = \langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \rangle$ gives the direction of maximum rate of change, and the maximum rate is $D_{\text{dir}(\nabla f)}f = |\nabla f| = 1$.

Since $f(x, y, z)$ is the distance from the origin to (x, y, z) , the answer makes sense geometrically: distance from the origin increases fastest when moving radially outwards, and the rate of increase is 1.

Problem 3.

a) $\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, -6 \rangle = \langle 8, 4, -6 \rangle$ at $(4, 2, 3)$. The direction of greatest decrease is that of $-\nabla g$, i.e. the unit vector $-\frac{\nabla g}{|\nabla g|} = \frac{\langle -8, -4, 6 \rangle}{\sqrt{116}} = \frac{\langle -4, -2, 3 \rangle}{\sqrt{29}}$.

b) Let $\Delta x = x - 4$, $\Delta y = y - 2$, $\Delta z = z - 3$; then the line in the direction of $\langle -4, -2, 3 \rangle$ can be parametrized by $\Delta x = -4t$, $\Delta y = -2t$, $\Delta z = 3t$. (Dividing by $\sqrt{29}$ is unnecessary and makes calculations more complicated.) At $P_0 = (4, 2, 3)$, we have $g = 2$, and $\nabla g = \langle 8, 4, -6 \rangle$ (from part (a)), so linear approximation gives

$$\begin{aligned} g(x, y, z) &\approx g(P_0) + \nabla g(P_0) \cdot \langle \Delta x, \Delta y, \Delta z \rangle = 2 + 8\Delta x + 4\Delta y - 6\Delta z \\ &= 2 + 8(-4t) + 4(-2t) - 6(3t) = 2 - 58t. \end{aligned}$$

Therefore, $g = 0$ when $2 - 58t \approx 0$, or $t \approx 1/29$. At $t = 1/29$, $(x, y, z) = (4 - 4t, 2 - 2t, 3 + 3t) = (4 - \frac{4}{29}, 2 - \frac{2}{29}, 3 + \frac{3}{29})$. Evaluating g at this point, we find ≈ 0.024 , fairly close to 0.

14.3 # 75: $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx$, and $u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin kx$, so indeed $u_t = \alpha^2 u_{xx}$.

14.3 # 81: by product rule and chain rule, $c(x, t) = (4\pi D)^{-1/2} t^{-1/2} e^{-x^2/(4Dt)} \Rightarrow$

$$\frac{\partial c}{\partial t} = -\frac{1}{2} \frac{t^{-3/2}}{(4\pi D)^{1/2}} e^{-x^2/(4Dt)} + \frac{x^2}{4Dt^2} (4\pi Dt)^{-1/2} e^{-x^2/(4Dt)} = \frac{-2Dt + x^2}{8\pi^{1/2} D^{3/2} t^{5/2}} e^{-x^2/(4Dt)},$$

$$\frac{\partial c}{\partial x} = \frac{-2x}{4Dt} (4\pi Dt)^{-1/2} e^{-x^2/(4Dt)} = \frac{-x}{4\pi^{1/2} D^{3/2} t^{3/2}} e^{-x^2/(4Dt)}, \quad \text{and}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{-1 + x(2x/4Dt)}{4\pi^{1/2} D^{3/2} t^{3/2}} e^{-x^2/(4Dt)} = \frac{-2Dt + x^2}{8\pi^{1/2} D^{5/2} t^{5/2}} e^{-x^2/(4Dt)}.$$

Comparing these expressions, we find that indeed $\partial c / \partial t = D \partial^2 c / \partial x^2$.

14.7 # 3: From the contour plot, there appears to be a local minimum near $(1, 1)$ (enclosed by oval-shaped level curves indicating that as we move away from the point in any direction the values of f are increasing). Moreover, the shape of the level curves near the origin is characteristic of a saddle point at $(0, 0)$.

To verify these guesses, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$. We have critical points when $f_x = f_y = 0$. The first equation $3x^2 - 3y = 0$ gives $y = x^2$, and substituting into the second equation gives $3(x^2)^2 - 3x = 0$, hence $3x(x^3 - 1) = 0$, hence $x = 0$ or $x = 1$. So we have two critical points $(0, 0)$ and $(1, 1)$.

The second partial derivatives are $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$, so $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$. Then $D(0, 0) = 0 - 9 < 0$ so f has a saddle point at $(0, 0)$; and $D(1, 1) = 36 - 9 > 0$, with $f_{xx}(1, 1) = 6 > 0$, so f has a local minimum at $(1, 1)$.

14.7 # 11: $f(x, y) = x^3 - 3x + 3xy^2 \Rightarrow f_x = 3x^2 + 3y^2 - 3$ and $f_y = 6xy$. Then $f_x = 0 \iff x^2 + y^2 = 1$, $f_y = 0 \iff xy = 0$.

Thus, at a critical point, either x or y is 0, and then the other one is ± 1 . There are four critical points: $(\pm 1, 0)$ and $(0, \pm 1)$.

Next, $f_{xx} = 6x$, $f_{xy} = 6y$, $f_{yy} = 6x$, so $D = f_{xx}f_{yy} - f_{xy}^2 = 36x^2 - 36y^2$.

$D(1, 0) = 36 > 0$ and $f_{xx}(1, 0) = 6 > 0$, so $(1, 0)$ is a local minimum.

$D(-1, 0) = 36 > 0$ and $f_{xx}(-1, 0) = -6 < 0$, so $(-1, 0)$ is a local maximum.

$D(0, \pm 1) = -36 < 0$, so $(0, 1)$ and $(0, -1)$ are saddle points.

14.7 # 43: We want to minimize the distance from $(4, 2, 0)$ to (x, y, z) , $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$, where $z^2 = x^2 + y^2$. Instead, it is easier to minimize $d^2 = f(x, y) = (x-4)^2 + (y-2)^2 + (x^2 + y^2)$. Since $f_x = 2(x-4) + 2x = 4x - 8$ and $f_y = 2(y-2) + 2y = 4y - 4$, the only critical point is $(x, y) = (2, 1)$. This point must correspond to the minimum distance ($f(x, y) \rightarrow \infty$ when x and/or y become large). For $x = 2$ and $y = 1$, the equation of the cone gives $z^2 = 5$ or $z = \pm\sqrt{5}$. Hence the points on the cone closest to $(4, 2, 0)$ are $(2, 1, \pm\sqrt{5})$.

14.7 # 49: Let (x, y, z) be the corner opposite the origin. Since $z = \frac{1}{3}(6 - x - 2y)$ and the volume is xyz , we want to maximize $f(x, y) = xyz = \frac{1}{3}xy(6 - x - 2y)$.

$f_x = \frac{1}{3}y(6 - 2x - 2y)$, and $f_y = \frac{1}{3}x(6 - x - 4y)$. Setting $f_x = f_y = 0$ gives $2x + 2y = 6$ and $x + 4y = 6$, hence the only critical point is $(x, y) = (2, 1)$, which geometrically must yield a maximum. Thus the volume of the largest box is $V = f(2, 1) = \frac{4}{3}$.