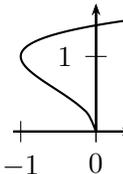


Math 53 Homework 3 – Solutions

10.2 # 7: (a) $x = 1 + \ln t$, $y = t^2 + 2$: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$. The point $(1, 3)$ is hit at $t = 1$, so $dy/dx = 2$, and the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(b) $t = e^{x-1}$, so $y = e^{2x-2} + 2$, and $dy/dx = 2e^{2x-2}$; when $x = 1$ we have $dy/dx = 2e^0 = 2$, and the equation of the tangent follows as in (a).

10.2 # 32: The curve $x = t^2 - 2t$, $y = \sqrt{t}$ lies to the left of the y -axis for $0 \leq t \leq 2$ ($x = t^2 - 2t \leq 0$), then crosses the y -axis; x decreases for $0 \leq t \leq 1$ then increases for $1 \leq t \leq 2$.



The area between the x -axis and the portion $0 \leq t \leq 1$ of the curve (which is traced from right to left) is $\int_1^0 y (dx/dt) dt$; while the area between the x -axis and the portion $1 \leq t \leq 2$ of the curve (which is traced from left to right) is $\int_1^2 y (dx/dt) dt$. Subtracting one from the other, the area we want is

$$\int_1^2 y (dx/dt) dt - \int_1^0 y (dx/dt) dt = \int_0^2 y (dx/dt) dt.$$

(So: in fact the signs work out exactly right and we can just integrate $y dx$ for $0 \leq t \leq 2$, despite the unusual setup). We calculate:

$$\int_0^2 y (dx/dt) dt = \int_0^2 \sqrt{t} (2t-2) dt = \int_0^2 (2t^{3/2} - 2t^{1/2}) dt = \left. \frac{4}{5}t^{5/2} - \frac{4}{3}t^{3/2} \right|_0^2 = \frac{8}{15}\sqrt{2}.$$

A more clever approach would have been to rotate the picture by 90 degrees clockwise, to reduce to the familiar situation of finding the area above the horizontal axis and below a curve. In other terms, the desired area is obtained by integrating $(-x) dy$ for $0 \leq t \leq 2$, i.e., $\int_0^2 (-x) (dy/dt) dt$, which gives

$$\int_0^2 -(t^2 - 2t) \frac{dt}{2\sqrt{t}} = \int_0^2 \left(t^{1/2} - \frac{1}{2}t^{3/2} \right) dt = \left. \frac{2}{3}t^{3/2} - \frac{1}{5}t^{5/2} \right|_0^2 = \frac{8}{15}\sqrt{2}.$$

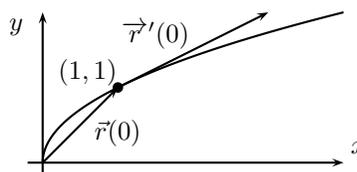
10.2 # 41: $dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$, and thus $L = \int_0^1 \sqrt{36t^2 + 36t^4} dt = \int_0^1 6t\sqrt{1+t^2} dt$. Substituting $u = 1+t^2$,

$$L = \int_0^1 6t\sqrt{1+t^2} dt = \int_1^2 3\sqrt{u} du = 2u^{3/2} \Big|_1^2 = 2(2\sqrt{2} - 1).$$

10.2 # 73: The coordinates of T (see figure in the book) are $(r \cos \theta, r \sin \theta)$. Since TP was unwound from the arc TA , TP has length $r\theta$. Also, TP is perpendicular to OT , so makes an angle $\theta - \frac{\pi}{2}$ with the x -axis (or θ with the negative y axis). So P has coordinates $x = r \cos \theta + r\theta \cos(\theta - \frac{\pi}{2}) = r \cos \theta + r\theta \sin \theta$ and $y = r \sin \theta + r\theta \sin(\theta - \frac{\pi}{2}) = r \sin \theta - r\theta \cos \theta$.

13.2 # 5: (a),(c) $x(t) = e^{2t}$, $y(t) = e^t$; eliminating t gives $y = \sqrt{x}$.

(b) $\vec{r}'(t) = 2e^{2t}\hat{i} + e^t\hat{j}$.



13.2 # 25: differentiating $\vec{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t, e^{-t} \rangle$, we get $\vec{r}'(t) = \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t} \rangle$.

The point $(1, 0, 1)$ corresponds to $t = 0$, and the tangent vector there is $\vec{r}'(0) = \langle -1, 1, -1 \rangle$. Thus, the tangent line is directed along the vector $\langle -1, 1, -1 \rangle$, and parametric equations are $x = 1 - t, y = t, z = 1 - t$.

13.2 # 33: $\vec{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$, while $\vec{r}_2'(t) = \langle \cos t, 2 \cos 2t, 1 \rangle$. Since both curves pass through the origin at $t = 0$, the tangent vectors there are respectively $\vec{r}_1'(0) = \langle 1, 0, 0 \rangle$ and $\vec{r}_2'(0) = \langle 1, 2, 1 \rangle$. The angle θ between these two vectors satisfies $\cos \theta = \frac{\langle 1, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle}{\sqrt{1^2 + 0^2 + 0^2} \sqrt{1^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{6}}$, and $\theta = \cos^{-1}(\frac{1}{\sqrt{6}}) \simeq 66^\circ$.

13.2 # 53: By eq. 5 of Theorem 3, $\frac{d}{dt}(\vec{r}(t) \times \vec{r}'(t)) = \vec{r}'(t) \times \vec{r}'(t) + \vec{r}(t) \times \vec{r}''(t)$. But $\vec{r}'(t) \times \vec{r}'(t) = \vec{0}$. So $\frac{d}{dt}(\vec{r}(t) \times \vec{r}'(t)) = \vec{r}(t) \times \vec{r}''(t)$.

13.2 # 55: $\frac{d}{dt}(|\vec{r}(t)|^2) = \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}(t) \cdot \vec{r}'(t)$. However, by the chain rule, $\frac{d}{dt}(|\vec{r}(t)|^2) = 2|\vec{r}(t)| \frac{d}{dt}|\vec{r}(t)|$.

So $\frac{d}{dt}|\vec{r}(t)| = \frac{1}{2|\vec{r}(t)|} \frac{d}{dt}(|\vec{r}(t)|^2) = \frac{1}{2|\vec{r}(t)|} 2\vec{r}(t) \cdot \vec{r}'(t) = \frac{1}{|\vec{r}(t)|} \vec{r}(t) \cdot \vec{r}'(t)$.

13.2 # 56: If $\vec{r}(t) \perp \vec{r}'(t)$, then $\vec{r}(t) \cdot \vec{r}'(t) = 0$. So, using the result of # 55, $\frac{d}{dt}|\vec{r}(t)| = 0$, and $|\vec{r}(t)|$ is constant. (Or: $0 = 2\vec{r}(t) \cdot \vec{r}'(t) = \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(|\vec{r}(t)|^2)$, so $|\vec{r}(t)|^2$ is constant and hence so is $|\vec{r}(t)|$.) So the trajectory remains at a constant distance from the origin, i.e. it lies on a sphere centered at the origin.

13.4 # 10: Velocity: $\vec{v}(t) = d\vec{r}/dt = \langle -2 \sin t, 3, 2 \cos t \rangle$.

Acceleration: $\vec{a}(t) = d\vec{v}/dt = \langle -2 \cos t, 0, -2 \sin t \rangle$.

Speed: $|\vec{v}(t)| = \sqrt{(-2 \sin t)^2 + 3^2 + (2 \cos t)^2} = \sqrt{4 + 9} = \sqrt{13}$ (independent of t).

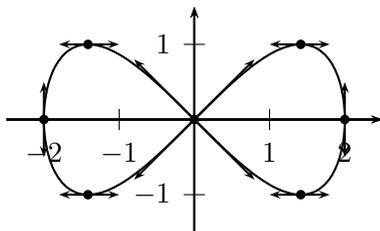
Problem 1. (a) $x = 2 \cos t, y = \sin 2t$: to find horizontal tangents, we compute $dy/dt = 2 \cos 2t$, so $dy/dt = 0$ if and only if $2t = \frac{\pi}{2} + n\pi$ (n integer), i.e. $t = \frac{\pi}{4} + n\frac{\pi}{2}$. This corresponds to the four points $(x, y) = (\pm\sqrt{2}, \pm 1)$ where the curve has horizontal tangencies.

Similarly, for vertical tangents, we find t for which $dx/dt = -2 \sin t = 0$, i.e. $t = n\pi$ (n integer), which gives the two points $(\pm 2, 0)$.

(b) $x = 0$ when $\cos t = 0$, i.e. $t = \pi/2 + n\pi$, and $y = 0$ when $\sin 2t = 0$, i.e. t is a multiple of $\pi/2$. Thus the point $(0, 0)$ is hit whenever $t = \pi/2 + n\pi$. This happens twice along the curve, at $t = \pi/2$ and $t = 3\pi/2$.

Recall from (a) that $dx/dt = -2 \sin t$ and $dy/dt = 2 \cos 2t$. At $t = \pi/2$, $dx/dt = -2$ and $dy/dt = -2$, so $dy/dx = 1$ and the tangent is $y = x$. When $t = 3\pi/2$, $dx/dt = 2$ and $dy/dt = -2$, so $dy/dx = -1$; the tangent is $y = -x$.

(c)



(d) The area enclosed in the first quadrant (between the x -axis and the portion of the curve corresponding to $0 \leq t \leq \pi/2$) is given by

$$\begin{aligned} \int_0^2 y \, dx &= \int_{\pi/2}^0 y \, (dx/dt) \, dt = \int_{\pi/2}^0 (\sin 2t) (-2 \sin t) \, dt = \int_0^{\pi/2} 2 \sin t \sin 2t \, dt \\ &= \int_0^{\pi/2} 2 \sin t (2 \sin t \cos t) \, dt = \int_0^{\pi/2} 4 \sin^2 t \cos t \, dt = \left[\frac{4}{3} \sin^3 t \right]_0^{\pi/2} = 4/3. \end{aligned}$$

Multiplying by 4 to account for all four quadrants, the total area enclosed is $16/3$.

Problem 2. (a) Let Q be the center of the rolling circle: since $|OQ| = 2a$ and the line OQ passes through the contact point, the coordinates of Q are $(2a \cos \theta, 2a \sin \theta)$.

To find the position of P , observe that $\angle OQP = \theta$: since one circle rolls on the other the arc lengths from S to R and from P to R are equal. So the line QP makes an angle of 2θ with the negative x -axis, and $|QP| = a$. So

$$\overrightarrow{QP} = \langle -a \cos 2\theta, -a \sin 2\theta \rangle.$$

Hence, $\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} = \langle 2a \cos \theta - a \cos 2\theta, 2a \sin \theta - a \sin 2\theta \rangle$, and the coordinates of P are:

$$x = 2a \cos \theta - a \cos 2\theta, \quad y = 2a \sin \theta - a \sin 2\theta.$$

(b) $dx/d\theta = 2a(-\sin \theta + \sin 2\theta)$, and $dy/d\theta = 2a(\cos \theta - \cos 2\theta)$, so

$$\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2 = 4a^2(\sin^2 \theta + \sin^2 2\theta - 2 \sin \theta \sin 2\theta + \cos^2 \theta + \cos^2 2\theta - 2 \cos \theta \cos 2\theta)$$

which simplifies (using $\sin^2 + \cos^2 = 1$) to $4a^2(2 - 2 \sin \theta \sin 2\theta - 2 \cos \theta \cos 2\theta) = 4a^2(2 - 2 \cos(2\theta - \theta)) = 4a^2(2 - 2 \cos \theta)$. So $L = \int_0^{2\pi} 2a\sqrt{2 - 2 \cos \theta} \, d\theta$.

Recall that $\cos \theta = 1 - 2 \sin^2(\theta/2)$, so $2 - 2 \cos \theta = 4 \sin^2(\theta/2)$,

and $L = \int_0^{2\pi} 4a \sin(\theta/2) \, d\theta = -8a \cos(\theta/2) \Big|_0^{2\pi} = -8a((-1) - 1) = 16a$.

Problem 3.

a) By symmetry, if the first bug is at $P = (x(t), y(t))$ then the second bug is equidistant from the origin, 90 degrees counterclockwise, hence at $Q = (-y(t), x(t))$.

Thus, the velocity of the first bug is $\vec{v} = \overrightarrow{PQ} = \langle -y(t) - x(t), x(t) - y(t) \rangle$.

b) $\frac{d}{dt}(|\vec{r}|^2) = \frac{d}{dt}(\vec{r} \cdot \vec{r}) = 2\vec{r} \cdot \vec{v} = 2\langle x, y \rangle \cdot \langle -y - x, x - y \rangle = -2(x^2 + y^2) = -2|\vec{r}|^2$.

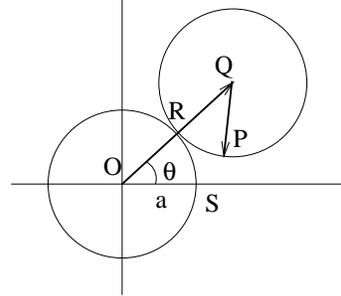
Hence, $f(t) = |\vec{r}(t)|^2$ is a solution of the differential equation $f'(t) = -2f(t)$, with $f(0) = 1$. Thus $|\vec{r}(t)|^2 = \exp(-2t)$, and $|\vec{r}| = e^{-t}$.

c) Since $\vec{r}(t)$ has length e^{-t} and makes an angle $\theta(t)$ with the x -axis, we have

$$\vec{r}(t) = \langle e^{-t} \cos \theta(t), e^{-t} \sin \theta(t) \rangle.$$

Differentiating, $\vec{v}(t) = \langle e^{-t}(-\cos \theta(t) - \frac{d\theta}{dt} \sin \theta(t)), e^{-t}(-\sin \theta(t) + \frac{d\theta}{dt} \cos \theta(t)) \rangle$.

We observe that $\vec{v}(t) = \langle -x(t) - \frac{d\theta}{dt}y(t), -y(t) + \frac{d\theta}{dt}x(t) \rangle$.



Comparing with the expression in (a), we find that $\frac{d\theta}{dt} = 1$. Hence $\theta(t) = t$, and

$$\vec{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t \rangle.$$

d) The velocity is $\vec{v}(t) = \langle e^{-t}(-\cos t - \sin t), e^{-t}(-\sin t + \cos t) \rangle$ (either by differentiating the formula for \vec{r} or by using (a)), and the speed is $|\vec{v}(t)| = \sqrt{2}e^{-t}$. (This can also be found directly using geometry: $|\vec{v}| = |\overrightarrow{PQ}| = \sqrt{2}|\overrightarrow{OP}|$.) Hence, the length of the trajectory is

$$L = \int_0^{\infty} |\vec{v}(t)| dt = \int_0^{\infty} \sqrt{2}e^{-t} dt = \sqrt{2}.$$

(Hence: the length of the trajectory is equal to the side length of the square formed by the initial positions of the four bugs.)