

## Math 53 Homework 2 – Solutions

**12.4 # 16:** (a)  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta = 3 \cdot 2 \cdot \sin\frac{\pi}{2} = 6$ .

(b)  $\vec{a} \times \vec{b}$  is orthogonal to  $\vec{b}$ , so it lies in the  $xy$ -plane and its  $z$ -component is 0. By the right-hand rule, its  $y$ -component is negative and its  $x$ -component is positive.

**12.4 # 31:** (a) The plane through  $P, Q, R$  contains the vectors  $\vec{PQ} = \langle 4, 3, -2 \rangle$  and  $\vec{PR} = \langle 5, 5, 1 \rangle$ , so a vector orthogonal to the plane is

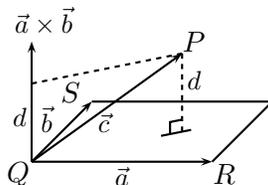
$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & -2 \\ 5 & 5 & 1 \end{vmatrix} = \langle 3 \cdot 1 - (-2) \cdot 5, (-2) \cdot 5 - 4 \cdot 1, 4 \cdot 5 - 3 \cdot 5 \rangle = \langle 13, -14, 5 \rangle$$

(or any multiple thereof).

(b) The area of the parallelogram determined by  $\vec{PQ}$  and  $\vec{PR}$  is  $|\vec{PQ} \times \vec{PR}| = |\langle 13, -14, 5 \rangle| = \sqrt{13^2 + (-14)^2 + 5^2} = \sqrt{390}$ , so the area of the triangle is  $\frac{1}{2}\sqrt{390}$ .

**12.4 # 36:**  $\det(\vec{PQ}, \vec{PR}, \vec{PS}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -36 + 8 + 44 = 16$ ; so the volume of the parallelepiped is 16.

**12.4 # 46:** (a)



Up to sign, the distance  $d$  from  $P$  to the plane is the component of  $\vec{c} = \vec{QP}$  along the direction perpendicular to the plane (see picture). Since  $\vec{a} = \vec{QR}$  and  $\vec{b} = \vec{QP}$  lie in the plane,  $\vec{a} \times \vec{b}$  is perpendicular to the plane. So the answer is (up to sign)

$$\text{comp}_{\vec{a} \times \vec{b}}(\vec{c}) = \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{|\vec{a} \times \vec{b}|} = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{a} \times \vec{b}|}.$$

This quantity may be positive or negative depending on which side of the plane  $P$  lies on, whereas distance is always measured positively, i.e.  $d = |\text{comp}_{\vec{a} \times \vec{b}}(\vec{c})|$ .

Alternative solution:  $d$  is the height of the parallelepiped with edges  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ . However, since the volume is the area of the base times the height, we get that  $d = (\text{volume})/(\text{area of base})$ . The volume is given by the absolute value of the determinant or triple product  $\vec{a} \cdot (\vec{b} \times \vec{c})$ . The area of the base (a parallelogram with sides  $\vec{a}$  and  $\vec{b}$ ) is given by  $|\vec{a} \times \vec{b}|$ . Dividing, we get the desired formula for  $d$ .

(b)  $\vec{a} = \vec{QR} = \langle -1, 2, 0 \rangle$ ,  $\vec{b} = \vec{QS} = \langle -1, 0, 3 \rangle$ , and  $\vec{c} = \vec{QP} = \langle 1, 1, 4 \rangle$ . Then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6, 3, 2 \rangle,$$

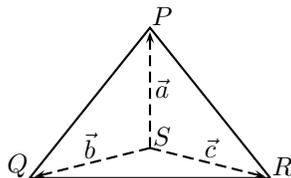
and  $(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 6, 3, 2 \rangle \cdot \langle 1, 1, 4 \rangle = 17$ , so  $d = \frac{|(\vec{a} \times \vec{b}) \cdot \vec{c}|}{|\vec{a} \times \vec{b}|} = \frac{17}{\sqrt{36 + 9 + 4}} = \frac{17}{7}$ .

**12.4 # 53:** (a) No.  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$  if and only if  $\vec{a} \cdot (\vec{b} - \vec{c}) = 0$ , which happens precisely when  $\vec{a}$  is perpendicular to  $\vec{b} - \vec{c}$ ; this can occur even if  $\vec{b} \neq \vec{c}$ .

(b) No.  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Leftrightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$ , which means that  $\vec{a}$  is parallel to  $\vec{b} - \vec{c}$ ; this can happen with  $\vec{b} \neq \vec{c}$ .

(c) Yes. If  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$  then  $\vec{a}$  is perpendicular to  $\vec{b} - \vec{c}$  by part (a). From part (b), if  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$  then  $\vec{a}$  is also parallel to  $\vec{b} - \vec{c}$ . Since  $\vec{a} \neq \vec{0}$  is both parallel and perpendicular to  $\vec{b} - \vec{c}$ , we must have  $\vec{b} - \vec{c} = \vec{0}$ , so  $\vec{b} = \vec{c}$ .

**p. 823 part 1:**



The vector coming out of the face opposite  $P$  (the bottom face) is  $\vec{v}_1 = \frac{1}{2} \overrightarrow{SR} \times \overrightarrow{SQ} = \frac{1}{2} \vec{c} \times \vec{b}$ . (Indeed, this vector is perpendicular to the face, its magnitude is equal to the area of the triangle, and by the right-hand rule it points downwards).

Similarly, for the face opposite  $Q$  we have  $\vec{v}_2 = \frac{1}{2} \overrightarrow{SP} \times \overrightarrow{SR} = \frac{1}{2} \vec{a} \times \vec{c}$ , and for the face opposite  $R$  we have  $\vec{v}_3 = \frac{1}{2} \overrightarrow{SQ} \times \overrightarrow{SP} = \frac{1}{2} \vec{b} \times \vec{a}$ .

Finally, for the face opposite  $S$  (the front face),  $\vec{v}_4 = \frac{1}{2} \overrightarrow{PQ} \times \overrightarrow{PR} = \frac{1}{2} (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$ .

Expanding and using the properties of cross product, we get:

$$\vec{v}_4 = \frac{1}{2} (\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}) = \frac{1}{2} (-\vec{c} \times \vec{b} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{0}) = -\vec{v}_1 - \vec{v}_3 - \vec{v}_2.$$

So  $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 = \vec{0}$ .

**part 3:** since the three edges meeting at the vertex  $S$  are mutually perpendicular, we can set up our coordinate system so that  $S$  is at the origin,  $SQ$  is the  $x$ -axis,  $SR$  is the  $y$ -axis,  $SP$  is the  $z$ -axis. Then the face opposite  $P$  lies in the  $xy$ -plane and has area  $A$ , so  $\vec{v}_1 = \langle 0, 0, -A \rangle$ . Similarly, the faces opposite  $Q$  and  $R$  lie in the  $yz$  and  $xz$  planes, and have areas  $B$  and  $C$ , so  $\vec{v}_2 = \langle -B, 0, 0 \rangle$  and  $\vec{v}_3 = \langle 0, -C, 0 \rangle$ . Using the result of part 1, we deduce that  $\vec{v}_4 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \langle B, C, A \rangle$ . Hence the area of the fourth face is  $D = |\vec{v}_4| = \sqrt{B^2 + C^2 + A^2}$ , so  $D^2 = B^2 + C^2 + A^2$ .

**12.5 # 5:** A line perpendicular to the given plane  $x + 3y + z = 5$  must be parallel to a normal vector to the plane, for instance  $\vec{n} = \langle 1, 3, 1 \rangle$ . So the vector equation is determined by the initial position  $\vec{r}_0 = \langle 1, 0, 6 \rangle$  and the direction vector  $\vec{v} = \vec{n} = \langle 1, 3, 1 \rangle$ , giving  $\vec{r} = \vec{r}_0 + t\vec{v} = \langle 1, 0, 6 \rangle + t\langle 1, 3, 1 \rangle = \langle 1+t, 3t, 6+t \rangle$ . Hence, parametric equations are:  $x = 1 + t$ ,  $y = 3t$ ,  $z = 6 + t$ .

**12.5 # 33:** Call the given points  $P, Q, R$ : the vectors  $\overrightarrow{PQ} = \langle 3-2, -8-1, 6-2 \rangle = \langle 1, -9, 4 \rangle$  and  $\overrightarrow{PR} = \langle -2-2, -3-1, 1-2 \rangle = \langle -4, -4, -1 \rangle$  lie in the plane, so a normal vector is  $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 25, -15, -40 \rangle$ . Hence, an equation of the plane is  $25(x-2) - 15(y-1) - 40(z-2) = 0$ , which simplifies to  $5x - 3y - 8z = -9$ .

**12.5 # 48:** A direction vector for the line through  $P_0 = (-3, 1, 0)$  and  $P_1 = (-1, 5, 6)$  is  $\vec{v} = \overrightarrow{P_0P_1} = \langle 2, 4, 6 \rangle$ , and parametric equations are  $x = -3 + 2t$ ,  $y = 1 + 4t$ ,  $z = 0 + 6t$ . Substituting these into the equation of the plane  $2x + y - z = -2$  gives  $2(-3 + 2t) + (1 + 4t) - 6t = -2$ , or  $-5 + 2t = -2$ , hence  $t = 3/2$ . Plugging  $t = 3/2$  into the parametric equations, we find that the intersection is at  $(0, 7, 9)$ .

**12.5 # 58:** (a) To find a point on the line of intersection, set one of the variables equal to a constant, say  $z = 0$ . (This will only work if the line of intersection crosses the  $xy$ -plane; otherwise try setting  $x$  or  $y$  equal to 0). Then the equations of the planes reduce to  $3x - 2y = 1$  and  $2x + y = 3$ . Solving these two equations gives  $x = 1$ ,  $y = 1$ . So a point on the line of intersection is  $(1, 1, 0)$ .

The direction of the line has to be contained in both planes, hence it should be perpendicular to both normal vectors  $\vec{n}_1 = \langle 3, -2, 1 \rangle$  and  $\vec{n}_2 = \langle 2, 1, -3 \rangle$ . Hence

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = \langle 5, 11, 7 \rangle \text{ is parallel to the line.}$$

This yields the parametric equations  $x = 1 + 5t$ ,  $y = 1 + 11t$ ,  $z = 7t$ . (There are many other parametric equations of the same line.)

(b) We find the angle between the normal vectors  $\vec{n}_1 = \langle 3, -2, 1 \rangle$  and  $\vec{n}_2 = \langle 2, 1, -3 \rangle$ :

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{6 - 2 - 3}{\sqrt{9 + 4 + 1} \sqrt{4 + 1 + 9}} = \frac{1}{14}. \text{ So } \theta = \cos^{-1}(1/14) \simeq 85.9^\circ.$$

**12.5 # 63:** The plane contains the points  $(a, 0, 0)$ ,  $(0, b, 0)$  and  $(0, 0, c)$ . Thus the vectors  $\vec{u} = \langle -a, b, 0 \rangle$  and  $\vec{v} = \langle -a, 0, c \rangle$  lie in the plane, and  $\vec{n} = \vec{u} \times \vec{v} = \langle bc, ac, ab \rangle$  is a normal vector to the plane. The equation of the plane is therefore  $bcx + acy + abz = abc$  (plugging in any of the given points to get the right hand side). If  $a, b, c$  are nonzero we can divide by  $abc$  and rewrite this equation as  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .

**12.5 # 77:**  $L_1 : x = y = z$ , and  $L_2 : x + 1 = y/2 = z/3$ ; so if a point  $(x, y, z)$  lies on both  $L_1$  and  $L_2$  then necessarily  $y = z$  and  $y/2 = z/3$ , which implies that  $y = z = 0$ ; however, considering  $L_1$  this implies that  $x = 0$ , while considering  $L_2$  we get  $x + 1 = 0$ , a contradiction. So the lines do not intersect. Moreover, the direction vectors are  $\vec{v}_1 = \langle 1, 1, 1 \rangle$  for  $L_1$  and  $\vec{v}_2 = \langle 1, 2, 3 \rangle$  for  $L_2$  (the denominators in the symmetric equations). So the lines are not parallel either; hence they are skew lines.

Since  $L_1$  and  $L_2$  are skew, they can be viewed as lying in two parallel planes, and the distance between the skew lines is the same as the distance between these planes. The common normal vector to the planes must be perpendicular to both  $\langle 1, 1, 1 \rangle$  and  $\langle 1, 2, 3 \rangle$  (the direction vectors of the two lines). So set  $\vec{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 3 - 2, -3 + 1, 2 - 1 \rangle = \langle 1, -2, 1 \rangle$ . From above, we have found that  $(0, 0, 0)$  and  $(-1, 0, 0)$  are points on  $L_1$  and  $L_2$  respectively. So equations of the planes  $P_1$  containing  $L_1$  and  $P_2$  containing  $L_2$  are respectively  $x - 2y + z = 0$  and  $x - 2y + z = -1$ .

The distance between the parallel planes  $P_1$  and  $P_2$  is the same as the distance of any point on  $P_2$  to  $P_1$ ; so for example we calculate the distance from the point  $(-1, 0, 0)$  to the plane  $x - 2y + z = 0$ . For this we use formula 9 on page 830:

$$D = \frac{|-1 - 2 \cdot 0 + 0|}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{1}{\sqrt{6}}.$$

*Alternative solution:* consider the points  $Q_1 = (0, 0, 0)$  on  $L_1$  and  $Q_2 = (-1, 0, 0)$  on  $L_2$ ; then the distance between the lines is the absolute value of the component of  $\vec{Q_1Q_2} = \langle -1, 0, 0 \rangle$  along the common normal vector  $\vec{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 1, -2, 1 \rangle$ . So  $D = \frac{|\vec{Q_1Q_2} \cdot \vec{n}|}{|\vec{n}|} = \frac{1}{\sqrt{6}}$ .

**Problem 1.** a) The line from  $E$  to  $P$  has the direction of  $\overrightarrow{EP} = \langle x_0 - 2, y_0, z_0 \rangle$  and goes through  $(2, 0, 0)$ , so it can be parametrized as 
$$\begin{cases} x = 2 + (x_0 - 2)t, \\ y = y_0t, \\ z = z_0t. \end{cases}$$

It intersects the  $yz$ -plane when  $x = 0$ , i.e.  $2 + (x_0 - 2)t = 0$ , which gives  $t = \frac{2}{2 - x_0}$ . So  $y = \frac{2y_0}{2 - x_0}$ ,  $z = \frac{2z_0}{2 - x_0}$ .

(We assume  $x_0 < 2$  because otherwise the point  $P$  would lie behind the observer).

b) The image on the screen of a line segment in space is contained in the intersection of the plane containing  $E$  and the line segment with the  $yz$ -plane. This intersection is a line, therefore the image is a line segment on the screen.

c) Using (a),  $(-1, -3, 1)$  is displayed at  $(-2, \frac{2}{3})$  in the  $yz$ -plane, and  $(-2, 4, 6)$  is displayed at  $(2, 3)$ , so the image on the screen is the line segment from  $(-2, \frac{2}{3})$  to  $(2, 3)$ .

d) The trajectory will again be a line segment. The velocity of the bird is  $\vec{v} = \overrightarrow{P_0P_1} = \langle -1, 7, 5 \rangle$ , so at time  $t$  its position is  $(-1 - t, -3 + 7t, 1 + 5t)$ . By the formula of part (a), this is displayed at

$$y = \frac{14t - 6}{t + 3} = \frac{14 - \frac{6}{t}}{1 + \frac{3}{t}}, \quad z = \frac{10t + 2}{t + 3} = \frac{10 + \frac{2}{t}}{1 + \frac{3}{t}}.$$

Taking the limit as  $t \rightarrow \infty$ , the position on the screen approaches  $(14, 10)$  (even though the actual position of the bird in 3D space is further and further away).

**10.1 # 13:** (a)  $x = \sin t$ ,  $y = 1/\sin t$ , so  $xy = 1$ .

(b) as  $t$  increases from 0 to  $\pi/2$ ,  $x$  increases from 0 to 1; hence, we get the portion of the hyperbola  $y = 1/x$  for  $0 < x < 1$ , traced from left to right (downwards).

**10.1 # 19:**  $(x - 5)^2 + (y - 3)^2 = 4 \cos^2 \pi t + 4 \sin^2 \pi t = 4$ , so the motion takes place on the circle of radius 2 centered at  $(5, 3)$ .

The portion where  $1 \leq t \leq 2$ , or  $\pi \leq \pi t \leq 2\pi$ , corresponds to the half-circle where  $\sin \pi t \leq 0$  hence  $y \leq 3$ , i.e. the bottom half of the circle; it is traced at constant speed, starting at  $(3, 3)$  for  $t = 1$  and ending at  $(7, 3)$  for  $t = 2$ , counterclockwise.

This can also be seen by starting from the parametrization of the unit circle  $x = \cos \pi t$ ,  $y = \sin \pi t$ , and observing that the given parametric curve differs from it by scaling  $x$  and  $y$  by a factor of 2 to get a circle of radius 2 and adding respectively 5 and 3 to move the center to  $(5, 3)$ .

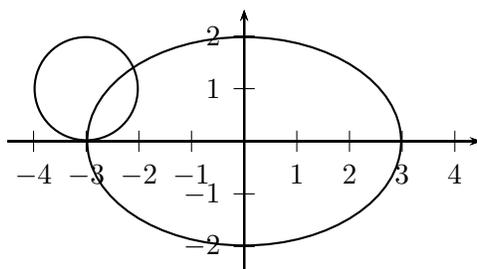
**10.1 # 33:** the circle of radius 2 centered at  $(0, 1)$  can be parametrized by  $x = 2 \cos t$ ,  $y = 1 + 2 \sin t$  where, as  $t$  varies from 0 to  $2\pi$ , the trajectory goes around the circle counterclockwise, starting at  $(2, 1)$ , and hitting  $(0, 3)$  at  $t = \pi/2$ . Hence:

(a) to get a clockwise orientation, we should change  $t$  to  $-t$ ; this yields  $x = 2 \cos t$ ,  $y = 1 - 2 \sin t$ ,  $0 \leq t \leq 2\pi$  (or any interval between consecutive multiples of  $2\pi$ ).

(b)  $x = 2 \cos t$ ,  $y = 1 + 2 \sin t$ ,  $0 \leq t \leq 6\pi$ .

(c)  $x = 2 \cos t$ ,  $y = 1 + 2 \sin t$ ,  $\pi/2 \leq t \leq 3\pi/2$ .

**10.1 # 45:** (a) There are two intersection points, one at  $(-3, 0)$  and the other near  $(-2.1, 1.4)$ .



(b) the intersection at  $(-3, 0)$  is a collision point, since it is hit by the first particle at  $t = 3\pi/2$  and by the second one at the same time  $t = 3\pi/2$ . On the other hand, the intersection near  $(-2.1, 1.4)$  is hit by the first particle (which moves clockwise on the large ellipse) at some time  $t_1$  with  $3\pi/2 < t_1 < 2\pi$ ; while the second particle passes through it for some time  $t_2$  with  $0 < t_2 < \pi/2$ ; since the particles are never there at the same time, it is not a collision point.

Or, more systematically: a collision point corresponds to  $t$  such that  $x_1(t) = x_2(t)$  and  $y_1(t) = y_2(t)$ , i.e.  $3\sin t = -3 + \cos t$  and  $2\cos t = 1 + \sin t$ . From the first equation we get that  $\cos t = 3 + 3\sin t$ , and plugging into the second equation we obtain  $5 + 5\sin t = 0$ ; this yields  $\sin t = -1$ , which corresponds to  $t = 3\pi/2$ , indeed a solution of both equations.

(c) the circle is now centered at  $(3, 1)$  instead of  $(-3, 1)$ . There are still two intersections, at  $(3, 0)$  and near  $(2.1, 1.4)$ ; but there are no collision points (for instance because the equations  $3\sin t = 3 + \cos t$  and  $2\cos t = 1 + \sin t$  imply that  $\sin t = 7/5$  and  $\cos t = 6/5$ , impossible.)

**13.1 # 14:** Since  $x = \cos t$  and  $y = -\cos t$ , the curve lies in the vertical plane  $y = -x$ . Moreover,  $y^2 + z^2 = \cos^2 t + \sin^2 t = 1$ , so the curve lies on the cylinder  $y^2 + z^2 = 1$  (or also on the cylinder  $x^2 + z^2 = 1$ ). It is therefore the ellipse where the cylinder and the plane intersect.

