Math 53 Homework 1 – Solutions

12.1 # 15: The radius of the sphere is the distance between (4,3,-1) and (3,8,1), namely $r = \sqrt{(3-4)^2 + (8-3)^2 + (1-(-1))^2} = \sqrt{30}$. Hence, an equation of the sphere is $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

12.1 # 37: $x^2 + z^2 \leq 9$: a cylinder of radius 3 centered on the y-axis. (The intersection of this solid in the xz-plane is the disk $x^2 + z^2 \leq 9$ of radius 3 centered at the origin; since the equation does not involve y, it intersects every plane parallel to the xz-plane in the same manner).

12.1 # 45: $P(x, y, z)$ satisfies $|AP| = |BP|$ if and only if

$$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}.$$

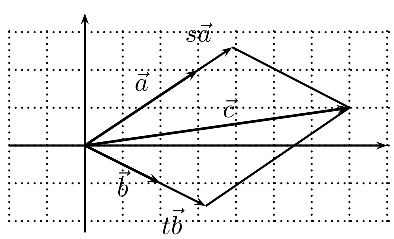
Squaring both sides and expanding, we get:

$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^4 + 4z + 4$, which simplifies to $14x - 6y - 10z = 9$. This is a plane – in fact, the plane perpendicular to the line segment AB through its midpoint, for symmetry reasons.

12.2 # 33: The two forces are given by the vectors $\vec{F_1} = \langle -300, 0\rangle$ and $\vec{F_2} = \langle 200 \cos 60^o, 200 \sin 60^o\rangle = \langle 100, 100\sqrt{3}\rangle$. The resultant force is $\vec{F} = \vec{F_1} + \vec{F_2} = \langle -300+100, 100\sqrt{3}\rangle = \langle -200, 100\sqrt{3}\rangle$. Its magnitude is $|\vec{F}| = \sqrt{(-200)^2 + (100\sqrt{3})^2} = 100\sqrt{4+3} = 100\sqrt{7} \simeq 264.6$ N.

The angle with the positive x-axis is determined by $\tan\theta = (100\sqrt{3})/(-200) = -\frac{\sqrt{3}}{2}$. $\tan^{-1}(-\sqrt{3}/2) \simeq -0.714$ radians (or -40.9^o). However, the vector points into the upper-left quadrant, so we must add π, and the angle is $\simeq 3.855$ radians or 139.1^o.

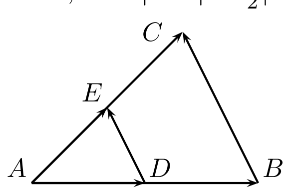
12.2 # 45: (a),(b)



(c) from the sketch, we estimate that $s \simeq 1.3$ and $t \simeq 1.6$.

(d) $s\vec{a} + t\vec{b} = \langle 3s + 2t, 2s - t\rangle$, so $s\vec{a} + t\vec{b} = \vec{c} \Leftrightarrow 3s + 2t = 7$ and $2s - t = 1$. Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

12.2 # 51: Consider the triangle ABC, and let D and E be the midpoints of AB and AC. Then $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$, and $\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC} - \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BC}$. Therefore $\overrightarrow{BC}$ and $\overrightarrow{DE}$ are parallel, and $|\overrightarrow{DE}| = \frac{1}{2}|\overrightarrow{BC}|$.



12.3 # 23: (a) $\vec{a}\cdot\vec{b} = (9)(-2) + (3)(6) = 0$, so $\vec{a}$ and $\vec{b}$ are orthogonal.
(b) $\vec{a}\cdot\vec{b} = (4)(3) + (5)(-1) + (-2)(5) = -3 \neq 0$, so $\vec{a}$ and $\vec{b}$ are not orthogonal. Also, $\vec{a}$ is not a scalar multiple of $\vec{b}$, so they are not parallel.
(c) $\vec{a} = -\frac{4}{3}\vec{b}$, so $\vec{a}$ and $\vec{b}$ are parallel.
(d) $\vec{a}\cdot\vec{b} = (3)(5) + (-1)(9) + (3)(-2) = 0$, so $\vec{a}$ and $\vec{b}$ are orthogonal.

12.3 # 25: $\overrightarrow{PQ} = \langle 1, 3, -2\rangle$, $\overrightarrow{PR} = \langle 5, 1, -3\rangle$, so $\overrightarrow{PQ}\cdot\overrightarrow{PR} = 5 + 3 + 6 = 14 \neq 0$, the angle at P isn't a right angle. (Similarly $\overrightarrow{RP}\cdot\overrightarrow{RQ} \neq 0$ so the angle at R isn't a right angle.) On the other hand, $\overrightarrow{QP} = \langle -1, -3, 2\rangle$, $\overrightarrow{QR} = \langle 4, -2, -1\rangle$, so $\overrightarrow{QP}\cdot\overrightarrow{QR} = -4 + 6 - 2 = 0$, hence $\overrightarrow{QR}\perp\overrightarrow{QP}$, there is a right angle at Q. So PQR is a right-angled triangle.

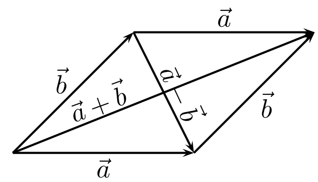
12.3 # 54: Using vectors:

$$\begin{aligned}
(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b}) &= \vec{r}\cdot\vec{r} - \vec{r}\cdot\vec{a} - \vec{r}\cdot\vec{b} + \vec{a}\cdot\vec{b} \\
&= (\vec{r} - \tfrac{1}{2}(\vec{a}+\vec{b}))\cdot(\vec{r} - \tfrac{1}{2}(\vec{a}+\vec{b})) - \tfrac{1}{4}(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + \vec{a}\cdot\vec{b} \\
&= |\vec{r} - \tfrac{1}{2}(\vec{a}+\vec{b})|^2 - \tfrac{1}{4}(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \\
&= |\vec{r} - \tfrac{1}{2}(\vec{a}+\vec{b})|^2 - |\tfrac{1}{2}(\vec{a}-\vec{b})|^2.
\end{aligned}$$

So $(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b}) = 0$ if and only if $|\vec{r} - \frac{1}{2}(\vec{a}+\vec{b})| = \frac{1}{2}|\vec{a}-\vec{b}|$.
This is a sphere centered at the point with coordinates given by the components of $\frac{1}{2}(\vec{a}+\vec{b})$, namely $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2})$ (the midpoint of the line segment connecting $A = (a_1, a_2, a_3)$ to $B = (b_1, b_2, b_3)$), and radius $\frac{1}{2}|\vec{a}-\vec{b}|$.
(One could also reach the same conclusion by working in components, starting from $(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b}) = (x-a_1)(x-b_1) + (y-a_2)(y-b_2) + (z-a_3)(z-b_3)$, and observing that $(x-a_1)(x-b_1) = x^2 - (a_1+b_1)x + a_1b_1 = (x - \frac{1}{2}(a_1+b_1))^2 - \frac{1}{4}(a_1+b_1)^2 + a_1b_1 = (x - \frac{1}{2}(a_1+b_1))^2 - \frac{1}{4}(a_1-b_1)^2$ and similarly for the other components. The calculation is essentially the same as above.)
Geometrically, the line segment connecting $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ is a diameter of this sphere. The property $(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b}) = 0$ says that the triangle with vertices A, B and $P = (x, y, z)$ has a right angle at P. This property of spheres is the analogue in 3d space of the classical geometric fact that an angle inscribed in a semicircle is a right angle.

12.3 # 55: Taking the unit cube with opposite vertices (0,0,0) and (1,1,1), an edge vector is $\vec{a} = \langle 1, 0, 0\rangle$, while a diagonal is $\vec{b} = \langle 1, 1, 1\rangle$. The angle θ between these vectors is given by $\cos\theta = \vec{a}\cdot\vec{b}/|\vec{a}|\,|\vec{b}| = \langle 1, 0, 0\rangle\cdot\langle 1, 1, 1\rangle/(1.\sqrt{3}) = 1/\sqrt{3}$. So $\theta = \cos^{-1}(1/\sqrt{3}) \simeq 0.955$ radians or 54.7 degrees. (Taking a different edge or a different cube would give the same answer.)

12.3 # 60: By assumption, the quadrilateral is a parallelogram, so its opposite sides are represented by the same vectors:



By assumption, $|\vec{a}| = |\vec{b}|$. The diagonals correspond to the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$; we compute $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b} = |\vec{a}|^2 - |\vec{b}|^2 = 0$. Hence the diagonals are perpendicular.

12.3 # 63: (a) $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ represent the diagonals of a parallelogram with sides $\vec{a}$ and $\vec{b}$ (see figure above).
So the identity $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$ states that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of its four sides.
(b) $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) =$
$= (\vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}) + (\vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}) = 2\vec{a}\cdot\vec{a} + 2\vec{b}\cdot\vec{b} = 2|\vec{a}|^2 + 2|\vec{b}|^2$.

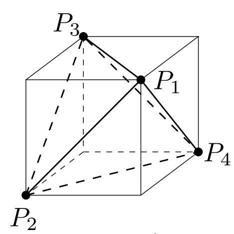
Problem 1. In components, $\vec{v}_1 = \langle \cos\theta_1, \sin\theta_1 \rangle$ and $\vec{v}_2 = \langle \cos\theta_2, \sin\theta_2 \rangle$, so

$$\vec{v}_1 \cdot \vec{v}_2 = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2.$$

Geometrically, $|\vec{v}_1| = |\vec{v}_2| = 1$, and the angle between the two vectors is $\theta_2 - \theta_1$, so $\vec{v}_1 \cdot \vec{v}_2 = \cos(\theta_2 - \theta_1)$. Comparing the two expressions we conclude that

$$\cos(\theta_2 - \theta_1) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2.$$

Problem 2. (a) $P_1 = (1,1,1)$, $P_2 = (1,-1,-1)$, $P_3 = (-1,-1,1)$, $P_4 = (-1,1,-1)$. Each pair of points differs by two sign changes from the others. All six edges of the tetrahedron are diagonals of faces of the cube and hence have the same length. For instance, $|\overrightarrow{P_1P_2}| = |\langle 1-1, -1-1, -1-1 \rangle| = |\langle 0,-2,-2 \rangle| = 2\sqrt{2}$.



(b) Adjacent edges: $\overrightarrow{P_1P_2} = \langle 0,-2,-2 \rangle$, $\overrightarrow{P_1P_3} = \langle -2,-2,0 \rangle$.

$$\cos\alpha = \frac{\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_3}}{|\overrightarrow{P_1P_2}||\overrightarrow{P_1P_3}|} = \frac{4}{(2\sqrt{2})^2} = \frac{1}{2}; \quad \alpha = \pi/3 = 60^\circ$$

The faces are equilateral triangles, so the angles are 60°.
Opposite edges: $\overrightarrow{P_1P_2} = \langle 0,-2,-2 \rangle$, $\overrightarrow{P_3P_4} = \langle 0,2,-2 \rangle$.

$$\cos\beta = \frac{\overrightarrow{P_1P_2} \cdot \overrightarrow{P_3P_4}}{|\overrightarrow{P_1P_2}||\overrightarrow{P_3P_4}|} = \frac{0}{(2\sqrt{2})^2} = 0; \quad \beta = \pi/2 = 90^\circ$$

By symmetry the perpendicular bisector to an edge contains the opposite edge, so these two edges are perpendicular to each other.
(c) $\cos\theta = \overrightarrow{OP_1} \cdot \overrightarrow{OP_2} / |\overrightarrow{OP_1}||\overrightarrow{OP_2}| = \langle 1,1,1 \rangle \cdot \langle 1,-1,-1 \rangle / (\sqrt{3})^2 = -1/3$. $\theta \approx 1.91$ radians (109.5°).