Math 53 Homework 13 – Solutions

16.9 # 17: Let S_1 be the disk $x^2+y^2 \le 1$ in the xy-plane, oriented downwards. Its normal vector is $\hat{\mathrm{n}} = -\hat{\mathrm{k}}$, so $\vec{F}\cdot\hat{\mathrm{n}} = -\vec{F}\cdot\hat{\mathrm{k}} = -(x^2z+y^2) = -y^2$ (since $z=0$ on S_1). Hence

$$\iint_{S_1} \vec{F}\cdot\hat{\mathrm{n}}\,dS = \iint_{S_1} -y^2\,dS = -\int_0^{2\pi}\int_0^1 (r\sin\theta)^2\,r\,dr\,d\theta.$$

Inner: $\left[\frac{1}{4}r^4\sin^2\theta\right]_0^1 = \frac{1}{4}\sin^2\theta$. Outer: $-\int_0^{2\pi}\frac{1}{4}\sin^2\theta\,d\theta = -\int_0^{2\pi}\frac{1}{8}(1-\cos 2\theta)\,d\theta = -\frac{\pi}{4}$.

Now, we apply the divergence theorem to the closed surface $S\cup S_1$. Observing that $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}(\frac{1}{3}y^3+\tan z) + \frac{\partial}{\partial z}(x^2z+y^2) = z^2+y^2+x^2$, we have:

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1}\vec{F}\cdot d\vec{S} = \iiint_E x^2+y^2+z^2\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= 2\pi\left(\int_0^{\pi/2}\sin\phi\,d\phi\right)\left(\int_0^1\rho^4\,d\rho\right) = (2\pi)(1)(\tfrac{1}{5}) = \tfrac{2}{5}\pi.$$

Finally, $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV - \iint_{S_1}\vec{F}\cdot d\vec{S} = \frac{2}{5}\pi - (-\frac{1}{4}\pi) = \frac{13}{20}\pi$.

16.9 # 19: The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 (or: the net flux out of a small disk around P_1 is negative), and so $\operatorname{div}\vec{F}$ is negative at P_1. Conversely, the vectors that end near P_2 are longer than those that start near P_2, so the net flow is outward near P_2 (or: the net flux out of a small disk around P_2 is positive), and so $\operatorname{div}\vec{F}$ is positive at P_2.

16.9 # 27: $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\operatorname{curl}\vec{F})\,dV$, however by Theorem 11 in 16.5 we have $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ (see section 16.5 for the proof), so $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iiint_E 0\,dV = 0$.

16.8 # 13: The boundary curve C is the circle of radius 4 in the plane $z=4$, $x^2+y^2=16$, oriented clockwise (since S is oriented downward), so it can be parametrized by $x=4\cos t$, $y=-4\sin t$, $z=4$, $0\le t\le 2\pi$. (The minus sign on y yields the clockwise orientation; alternatively one could keep the usual sign but run backwards from $t=2\pi$ to $t=0$.) Thus

$$\int_C \vec{F}\cdot d\vec{r} = \int_C -y\,dx + x\,dy - 2\,dz$$
$$= \int_0^{2\pi}(4\sin t)(-4\sin t)\,dt + (4\cos t)(-4\cos t)\,dt + 0\,dt = \int_0^{2\pi} -16\,dt = -32\pi.$$

S is the graph of $f(x,y) = \sqrt{x^2+y^2}$ over the disc $x^2+y^2\le 16$, oriented downwards, so $\hat{\mathrm{n}}\,dS = \langle f_x, f_y, -1\rangle\,dx\,dy = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\rangle\,dx\,dy$.

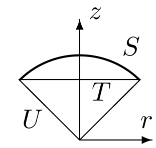
Since $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathrm{k}} \\ \partial_x & \partial_y & \partial_z \\ -y & x & -2 \end{vmatrix} = 2\hat{\mathrm{k}}$, we find $\operatorname{curl}\vec{F}\cdot\hat{\mathrm{n}}\,dS = -2\,dx\,dy$.

Hence $\iint_S \operatorname{curl}\vec{F}\cdot\hat{\mathrm{n}}\,dS = \iint_S -2\,dx\,dy = \int_0^{2\pi}\int_0^4 -2\,r\,dr\,d\theta = (2\pi)(-16) = -32\pi$, same as the direct calculation of $\oint_C \vec{F}\cdot d\vec{r}$.

16.8 # 15: The boundary curve C is the unit circle in the xz-plane, $x^2+z^2=1$, $y=0$, oriented counterclockwise as seen in the usual projection; it can be parametrized by $x=\sin t$, $y=0$, $z=\cos t$ (or equivalently $x=\cos t$, $y=0$, $z=-\sin t$; check that the orientation is as claimed!). So $\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + z\,dy + x\,dz = \int_C x\,dz = \int_0^{2\pi} -\sin^2 t\,dt = -\pi$.

Meanwhile, $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathrm{k}} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = \langle -1,-1,-1\rangle$, and on the unit hemisphere $\hat{\mathrm{n}} = \langle x,y,z\rangle$, so $\operatorname{curl}\vec{F}\cdot\hat{\mathrm{n}} = -(x+y+z)$. Parametrizing S by $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$, $z=\cos\phi$, $0\le\phi\le\pi$, $0\le\theta\le\pi$, we get

$\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS = \iint_S -(x+y+z)\,dS = \int_0^\pi\int_0^\pi -(\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi)\,\sin\phi\,d\phi\,d\theta.$
Inner: $\int_0^\pi -\sin^2\phi(\cos\theta+\sin\theta)-\cos\phi\sin\phi\,d\phi = -\frac{\pi}{2}(\cos\theta+\sin\theta).$
Outer: $\int_0^\pi -\frac{\pi}{2}(\cos\theta+\sin\theta)\,d\theta = -\frac{\pi}{2}\left[\sin\theta-\cos\theta\right]_0^\pi = -\pi.$ (Same as $\int_C \vec{F}\cdot d\vec{r}$.)



Problem 1. Radius of disk T: intersection of $z^2 = x^2+y^2$ (cone) and $x^2+y^2+z^2 = 2$ (sphere). By elimination, we get $x^2+y^2 = z^2 = 1$, i.e. $z = 1$ and radius $r = 1$.

a) $\vec{F} = x\hat{\imath}+y\hat{\jmath}$ is horizontal and points radially outwards (away from the z-axis). Therefore, the flux across S is positive ($\vec{F}$ points out of the sphere); the flux across T is zero ($\vec{F}$ is parallel to the horizontal plane containing T); the flux across U is negative ($\vec{F}$ points out of the cone, while the normal vector points up and into the cone).

b) Across S (spherical cap, $\rho = \sqrt{2}$, $\phi < \frac{\pi}{4}$): $dS = \rho^2\sin\phi\,d\phi\,d\theta = 2\sin\phi\,d\phi\,d\theta$.
Unit normal $\hat{n} = \frac{1}{\rho}(x\hat{\imath}+y\hat{\jmath}+z\hat{k})$, hence $\vec{F}\cdot\hat{n} = \frac{1}{\rho}(x^2+y^2) = \rho\sin^2\phi = \sqrt{2}\sin^2\phi$, and

$$\begin{aligned}\iint_S \vec{F}\cdot\hat{n}\,dS &= \int_0^{2\pi}\int_0^{\pi/4}(\sqrt{2}\sin^2\phi)\,(2\sin\phi)\,d\phi\,d\theta = 2\sqrt{2}\,(2\pi)\int_0^{\pi/4}\sin^3\phi\,d\phi\\ &= 4\pi\sqrt{2}\int_0^{\pi/4}\sin\phi(1-\cos^2\phi)\,d\phi = 4\pi\sqrt{2}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/4}\\ &= 4\pi\sqrt{2}\Big[(-\frac{1}{\sqrt{2}}+\frac{1}{6\sqrt{2}})-(-1+\frac{1}{3})\Big] = \frac{(8\sqrt{2}-10)\pi}{3}.\end{aligned}$$

Across T: $\hat{n} = \hat{k}$, so $\vec{F}\cdot\hat{n} = 0$ and $\iint_T \vec{F}\cdot\hat{n}\,dS = 0$.
Across U (cone, graph of $f(x,y) = \sqrt{x^2+y^2}$ over unit disk): $\hat{n}\,dS = \langle -f_x,-f_y,1\rangle\,dA = \langle -x/r,-y/r,1\rangle\,dA$, so $\vec{F}\cdot\hat{n}\,dS = \langle x,y,0\rangle\cdot\langle -x/r,-y/r,1\rangle\,dA = (-r)\,(r\,dr\,d\theta)$.

$$\iint_U \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^1 -r^2\,dr\,d\theta = -(2\pi)\,\frac{1}{3} = -\frac{2\pi}{3}.$$

c) $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(0) = 2$. Therefore, the flux out of the solid cone D_1 is

$\iint \vec{F}\cdot\hat{n}\,dS = \iiint_{D_1} 2\,dV = 2\,\text{volume}(D_1) = 2\,(\frac{1}{3}\,\pi) = \frac{2\pi}{3}.$ (volume = base × height /3).

Flux out of the region D_2 bounded by S and U:

$$\begin{aligned}\iint \vec{F}\cdot\hat{n}\,dS &= \iiint_{D_2} 2\,dV = 2\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= 2\,(2\pi)\int_0^{\pi/4}\frac{1}{3}\rho^3\sin\phi\Big|_0^{\sqrt{2}}\,d\phi\\ &= 4\pi\,\frac{2\sqrt{2}}{3}\int_0^{\pi/4}\sin\phi\,d\phi = \frac{8\pi\sqrt{2}}{3}(-\frac{1}{\sqrt{2}}-(-1)) = \frac{8(\sqrt{2}-1)\pi}{3}.\end{aligned}$$

d) Recall from part (b): taking normal vectors pointing *up*, the flux through the spherical cap S is $(8\sqrt{2}-10)\pi/3$; the flux through the disk T is 0; through the cone U it is $-2\pi/3$. The oriented boundary of the solid cone is $T-U$ (normal vectors should point out of the cone, which agrees with our previous choice for T but *not* for U). From the direct calculation in part (b), $\iint_{T-U}\vec{F}\cdot\hat{n}\,dS = \iint_T - \iint_U = 0+\frac{2\pi}{3} = \frac{2\pi}{3}$, which agrees with (c).

Similarly, the oriented boundary of D_2 is $S-U$. From part (b), we have
$$\iint_{S-U} \vec F\cdot\hat{\mathrm n}\,dS = \iint_S - \iint_U = \frac{(8\sqrt2-10)\pi}{3}+\frac{2\pi}{3} = \frac{(8\sqrt2-8)\pi}{3},$$ in agreement with (c).

Problem 2.

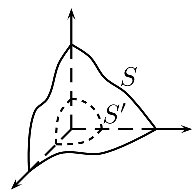
a) $\vec F = \dfrac{-x\,\hat\imath - y\,\hat\jmath - z\,\hat{\mathrm k}}{(x^2+y^2+z^2)^{3/2}} = -\dfrac{x\,\hat\imath + y\,\hat\jmath + z\,\hat{\mathrm k}}{\rho^3}$ is directed radially inward, with length $1/\rho^2$.

b) From the geometric description, $\vec F\cdot\hat{\mathrm n} = -1/\rho^2 = -1/a^2$ on the sphere $\rho=a$. Therefore $\iint_S \vec F\cdot\hat{\mathrm n}\,dS = -\frac{1}{a^2}\iint_S dS = -\frac{1}{a^2}4\pi a^2 = -4\pi$.

c) $\frac{\partial}{\partial x}(-x\rho^{-3}) = -\rho^{-3} - x\cdot(-3\,\rho_x\,\rho^{-4}) = -\rho^{-3}+3x^2\rho^{-5}$ (using $\rho_x = x/\rho$); similarly for y and z. Therefore, $\operatorname{div}\vec F = -3\rho^{-3} + 3(x^2+y^2+z^2)\rho^{-5} = -3\rho^{-3}+3\rho^{-3} = 0$.

The divergence theorem cannot be used to compute the flux of $\vec F$ over the sphere $\rho = a$, because $\vec F$ is not defined at every point of the interior ball ($\vec F$ is not defined at the origin). So there is no contradiction.

d) Consider S' = the portion of a small sphere centered at the origin which lies in the first octant, oriented outwards, and let D be the portion of the first octant between S' and the given surface S. The flux of $\vec F$ outwards through S' (into D) is 1/8 of that through the entire sphere (by symmetry), i.e., using the result of (b), $-4\pi/8 = -\pi/2$.



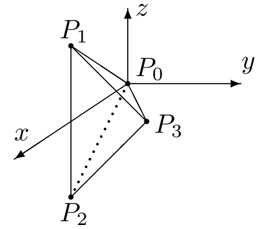
The boundary of D consists of S, $-S'$, and three flat "sides" which are portions of the coordinate planes. Because $\vec F$ points radially towards the origin, it is tangent to the coordinate planes, and the flux through the sides is zero. Moreover, $\operatorname{div}\vec F = 0$ by the result of (c), so by the divergence theorem the total flux of $\vec F$ out of D is zero. So:
$$0 = \iint_S \vec F\cdot\hat{\mathrm n}\,dS - \iint_{S'} \vec F\cdot\hat{\mathrm n}\,dS + \iint_{\text{sides}} \vec F\cdot\hat{\mathrm n}\,dS = \iint_S \vec F\cdot\hat{\mathrm n}\,dS - (-\pi/2) + 0.$$
Hence $\iint_S \vec F\cdot\hat{\mathrm n}\,dS = -\pi/2$.

Problem 3. a) $\operatorname{curl}\vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat{\mathrm k} \\ \partial_x & \partial_y & \partial_z \\ -2xz & 0 & y^2 \end{vmatrix} = 2y\hat\imath - 2x\hat\jmath.$

b) On the unit sphere, the normal vector is $\hat{\mathrm n} = x\hat\imath + y\hat\jmath + z\hat{\mathrm k}$, so the integrand in the flux of $\operatorname{curl}\vec F$ is $\operatorname{curl}\vec F\cdot\hat{\mathrm n} = \langle 2y,-2x,0\rangle\cdot\langle x,y,z\rangle = 2xy-2xy = 0$. Therefore, let C be a simple closed curve on the unit sphere, and let S be the portion of the surface of the sphere delimited by C. Then by Stokes' theorem, $\oint_C \vec F\cdot d\vec r = \iint_S \operatorname{curl}\vec F\cdot\hat{\mathrm n}\,dS = \iint_S 0\,dS = 0$.

Problem 4.

a) For $P_0P_1P_2$: from P_0 to P_2 to P_1 back to P_0. For $P_0P_1P_3$: from P_0 to P_1 to P_3 back to P_0. For $P_0P_2P_3$: from P_0 to P_3 to P_2 back to P_0. For $P_1P_2P_3$: P_1 to P_2 to P_3 back to P_1.



b) From P_0 to P_1: $x=t$, $y=0$, $z=t$ for $0\le t\le 1$, so
$$\int_{P_0P_1} yz\,dy - y^2\,dz = \int_0^1 0\,dt = 0.$$
From P_1 to P_3: $x=1$, $y=t$, $z=1-t$ for $0\le t\le 1$, so
$$\int_{P_1P_3} yz\,dy - y^2\,dz = \int_0^1 t(1-t)\,dt - t^2\,(-dt) = \int_0^1 t\,dt = \frac12.$$

From P_3 to P_0: $x = t$, $y = t$, $z = 0$ for t going from 1 to 0, so $\displaystyle\int_{P_3P_0} yz\,dy - y^2\,dz = \int_1^0 0\,dt = 0$.

Therefore, the total work is $0 + \dfrac{1}{2} + 0 = \dfrac{1}{2}$.

c) $\nabla \times \vec{F} = (\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(yz))\hat{\imath} = -3y\hat{\imath}$, so by Stokes' theorem, for each face we have $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,dS = \iint_S (-3y\hat{\imath}) \cdot \hat{n}\,dS$. Hence we have to find the flux of the vector field $\vec{G} = \nabla \times \vec{F} = -3y\hat{\imath}$ through each face.

Through $P_0P_1P_2$: the face is contained in the xz-plane ($y = 0$), so the outward unit normal is $\hat{n} = -\hat{\jmath}$. Since $\vec{G} \cdot \hat{n} = 0$, the flux is zero.

Through $P_1P_2P_3$: the face is contained in the plane $x = 1$, so the outward unit normal is $\hat{n} = \hat{\imath}$, and $\vec{G} \cdot \hat{n} = \langle -3y, 0, 0\rangle \cdot \langle 1, 0, 0\rangle = -3y$.

$$\iint_{P_1P_2P_3} \vec{G}\cdot\hat{n}\,dS = \iint_{P_1P_2P_3} -3y\,dS = \int_0^1 \int_{-(1-y)}^{1-y} -3y\,dz\,dy = \int_0^1 -6y(1-y)\,dy = \Big[-3y^2 + 2y^3\Big]_0^1 = -1.$$

Through $P_0P_1P_3$: a normal vector (pointing outwards) is $\vec{N} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_3} = \langle -1, 1, 1\rangle$, so $P_0P_1P_3$ is contained in the plane $-x + y + z = 0$, i.e. the graph $z = x - y$ of $f(x, y) = x - y$; so $\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy = \langle -1, 1, 1\rangle\,dx\,dy$.

So $\vec{G} \cdot \hat{n}\,dS = \langle -3y, 0, 0\rangle \cdot \langle -1, 1, 1\rangle\,dx\,dy = 3y\,dx\,dy$. The projection of the face $P_0P_1P_3$ on the xy-plane is a triangle with vertices at $(0, 0)$, $(1, 0)$ and $(1, 1)$, so

$$\iint_{P_0P_1P_3} \vec{G} \cdot \hat{n}\,dS = \int_0^1 \int_0^x 3y\,dy\,dx = \int_0^1 \frac{3}{2}x^2\,dx = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2}.$$

The symmetry $(x, y, z) \longrightarrow (x, y, -z)$ exchanges the two faces $P_0P_1P_3$ and $P_0P_2P_3$, so the two normal vectors are symmetric to each other (the orientations match). Since $\vec{G} = -3y\hat{\imath}$ is also preserved by this symmetry, the flux through $P_0P_2P_3$ is the same as through $P_0P_1P_3$, namely $1/2$.

(Or: $P_0P_2P_3$ is contained in the plane $z = -x + y$, so $\hat{n}\,dS = -\langle 1, -1, 1\rangle\,dx\,dy$ (the negative sign is so $\hat{n}$ points downwards) and $\vec{G} \cdot \hat{n}\,dS = 3y\,dx\,dy$. The projection of the face onto the xy-plane is again the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$, and the calculation proceeds as previously to give $1/2$.)

d) (i) When we add together the four answers from (c), we compute work along a curve that passes twice over each of the 6 edges of the tetrahedron. However each edge is traversed once with each orientation, so the various contributions cancel each other. For example the edge P_0P_1 is encountered once for the face $P_0P_1P_2$ (it is then oriented from P_1 to P_0) and once for the face $P_0P_1P_3$ (it is then oriented from P_0 to P_1); the sum of the two contributions is zero.

(ii) For each face $\displaystyle\oint F \cdot d\vec{r} = \iint (\operatorname{curl} \vec{F}) \cdot \hat{n}\,dS$ by Stokes, so the sum of the four line integrals is the flux of $\operatorname{curl} \vec{F} = -3y\hat{\imath}$ out of the tetrahedron. By the divergence theorem, $\displaystyle\iint_S (\operatorname{curl} \vec{F}) \cdot \hat{n}\,dS = \iiint_D \operatorname{div}(\operatorname{curl} \vec{F})\,dV$. But $\operatorname{div}(\operatorname{curl} \vec{F}) = \operatorname{div}(-3y\hat{\imath}) = 0$, so the total flux is zero.