

Math 53 Homework 13

Due **Wednesday 11/29/17** in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 11/13: Divergence theorem continued; Review

- **Read:** section 16.9.
- **Work:** 16.9: 17, 19, 27, (29).
 Problems 1 and 2 below.

Wednesday 11/15: MIDTERM 2

Friday 11/17: Stokes' theorem

- **Read:** section 16.8.
- **Work:** 16.8: (1), (3), (5), (7), (9), 13¹, 15.
 Problems 3 and 4 below.

Monday 11/20: More on partial differential equations

Monday 11/27: More on divergence and Stokes

- **Read:** sections 16.8 and 16.10.

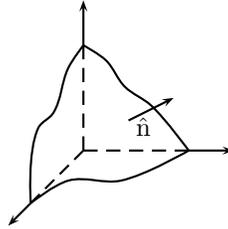
Problem 1. Consider the space region bounded below by the right-angled cone $z = \sqrt{x^2 + y^2}$, and above by the sphere $x^2 + y^2 + z^2 = 2$. These two surfaces intersect in a horizontal circle; let T be the horizontal disk having this circle as boundary, S the spherical cap forming the upper surface, and U the cone forming the lower surface. Orient S , T , U “upwards”, so the normal vector has a positive \hat{k} -component.

- For each of the three surfaces, determine geometrically (without calculation) whether the flux of the vector field $\vec{F} = x\hat{i} + y\hat{j}$ is positive or negative.
- Calculate the flux of \vec{F} across each surface (with the upwards orientation). (Do not use the divergence theorem).
- Use the divergence theorem to find the flux of $\vec{F} = x\hat{i} + y\hat{j}$ out of the solid cone bounded by T and U . Same question with the ice-cream cone bounded by S and U .
- Show that the answers you found in part (c) are consistent with those you found in part (b). (Be careful with orientations!)

¹**7th ed:** do the 8th ed problem: $\vec{F}(x, y, z) = -y\hat{i} + x\hat{j} - 2\hat{k}$, S is the cone $z^2 = x^2 + y^2$, $0 \leq z \leq 4$, oriented downwards.

Problem 2.

- a) Let $f(x, y, z) = 1/\rho = (x^2 + y^2 + z^2)^{-1/2}$. Calculate $\vec{F} = \nabla f$, and describe geometrically the vector field \vec{F} .
- b) Evaluate the flux of \vec{F} over the sphere of radius a centered at the origin.
- c) Show that $\text{div } \vec{F} = 0$. Does the answer obtained in (b) contradict the divergence theorem? Explain.
- d) Let S be a surface in the first octant, whose boundary lies in the three coordinate planes (see picture). Show that $\iint_S \vec{F} \cdot \hat{n} dS$ is independent of the choice of S , and calculate its value. (Hint: apply the divergence theorem to a suitable portion of the first octant).

**Problem 3.**

- a) Calculate the curl of $\vec{F} = -2xz\hat{i} + y^2\hat{k}$.
- b) Using Stokes' theorem, show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C drawn on the unit sphere $x^2 + y^2 + z^2 = 1$.

Problem 4.

Consider the tetrahedron with vertices at $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$, and $P_3 = (1, 1, 0)$.

- a) Say which orientation (order of vertices) of the boundary curve of each face is compatible with the choice of the normal vector pointing out of the tetrahedron.
- b) Compute the work done by the vector field $\vec{F} = yz\hat{j} - y^2\hat{k}$ around the boundary curve of the face $P_0P_1P_3$ directly using line integrals (using the orientation from part (a)).
- c) Use Stokes' theorem to compute the work done around each of the four faces (including the one you computed directly in part (b)). Use the orientations from part (a).

(Note: the symmetry $z \rightarrow -z$ exchanges two of the faces of the tetrahedron, and can be used to avoid one calculation – if you choose to use symmetry, you need to explain why it is legitimate.)

- d) The sum of the four values you found in part (c) should be zero. Explain this in two different ways:
- (i) geometrically, by considering the various line integrals that are being added together;
- (ii) by using the divergence theorem to compute the flux of $\text{curl } \vec{F}$ out of the tetrahedron.