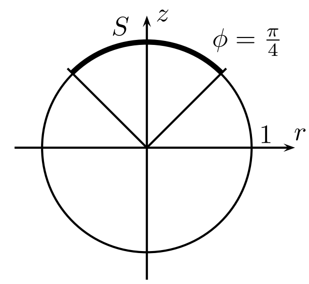
Math 53 Homework 12 – Solutions

16.7 # 16: In terms of the spherical angles ϕ and θ, we have $dS = 1^2 \sin\phi \, d\phi \, d\theta$ (cf. Example 10 in 16.6; the radius of the sphere is 1); and $y^2 = (\sin\phi \sin\theta)^2$. The portion of sphere we are considering corresponds to $0 \le \phi \le \pi/4$ (see picture). Hence



$$\iint_S y^2 \, dS = \int_0^{2\pi} \int_0^{\pi/4} (\sin\phi \sin\theta)^2 \sin\phi \, d\phi \, d\theta = \left(\int_0^{\pi/4} \sin^3\phi \, d\phi \right) \left(\int_0^{2\pi} \sin^2\theta \, d\theta \right).$$

$$\int_0^{\pi/4} \sin^3\phi \, d\phi = \int_0^{\pi/4} (1 - \cos^2\phi) \sin\phi \, d\phi = \left[-\cos\phi + \frac{1}{3}\cos^3\phi \right]_0^{\pi/4} = \frac{2}{3} - \frac{5\sqrt{2}}{12}, \text{ and}$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta) \, d\theta = \pi, \text{ so } \iint_S y^2 \, dS = \left(\frac{2}{3} - \frac{5\sqrt{2}}{12} \right) \pi.$$

16.7 # 18: Parametrize the cylinder by $y = y$, $x = \cos t$, $y = y$, $z = \sin t$: then $dS = dy \, dt$ (either by calculating $|\vec{r}_y \times \vec{r}_t| = |\langle \cos t, 0, \sin t \rangle| = 1$ or by geometry). The surface S corresponds to $0 \le y \le 2$ and $0 \le t \le \pi$. Then

$$\iint_S (x+y+z) \, dS = \int_0^2 \int_0^\pi (\cos t + y + \sin t) \, dt \, dy = \int_0^2 (\pi y + 2) \, dy = \left[\frac{1}{2}\pi y^2 + 2y \right]_0^2 = 2\pi + 4.$$

16.7 # 23: $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$, and since $z = g(x,y) = 4 - x^2 - y^2$, for the upward orientation we have $d\vec{S} = \langle -g_x, -g_y, 1 \rangle \, dx \, dy = \langle 2x, 2y, 1 \rangle \, dx \, dy$. Hence

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (2x^2y + 2y^2z + xz) \, dx \, dy = \int_0^1 \int_0^1 (2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2)) \, dy \, dx.$

Inner: $\int_0^1 -2y^4 + (8 - 2x^2 - x)y^2 + 2x^2y + (4x - x^3) \, dy = -\frac{2}{5} + \frac{1}{3}(8 - 2x^2 - x) + x^2 + (4x - x^3)$.

Outer: $\int_0^1 (-x^3 + \frac{1}{3}x^2 + \frac{11}{3}x + \frac{34}{15}) \, dx = -\frac{1}{4} + \frac{1}{9} + \frac{11}{6} + \frac{34}{15} = \frac{713}{180}$.

16.7 # 24: $\vec{F} = \langle -x, -y, z^3 \rangle$, and since $z = g(x,y) = \sqrt{x^2 + y^2}$, with the downward orientation we have $d\vec{S} = -\langle -g_x, -g_y, 1 \rangle \, dx \, dy = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \rangle \, dx \, dy$.

Moreover, the surface S lies above the annulus $1 \le \sqrt{x^2+y^2} \le 3$ in the $xy-$plane (since the cone intersects $z = 1$ and $z = 3$ along circles of radii 1 and 3). Hence

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \left(\frac{-x^2 - y^2}{\sqrt{x^2+y^2}} - z^3 \right) dx \, dy = \iint_S (-r - z^3) \, r \, dr \, d\theta = \int_0^{2\pi} \int_1^3 (-r - r^3) \, r \, dr \, d\theta$$

(using: $z = r$ everywhere on S). Inner: $\left[-\frac{1}{3}r^3 - \frac{1}{5}r^5 \right]_1^3 = -\frac{1}{3}(27 - 1) - \frac{1}{5}(243 - 1) = -\frac{856}{15}$.
Outer: $2\pi \cdot \frac{-856}{15} = -1712\pi/15$.

16.7 # 26: The normal vector points radially inwards, i.e. it is negatively proportional to $\langle x, y, z \rangle$. Since $|\langle x, y, z \rangle| = \sqrt{x^2 + y^2 + z^2} = 2$ on S, we have $\hat{n} = -\frac{1}{2}\langle x, y, z \rangle$. Hence $\vec{F} \cdot \hat{n} = \langle y, -x, 2z \rangle \cdot (-\frac{1}{2}\langle x, y, z \rangle) = -z^2$.

We parametrize the spherical surface using the angles ϕ and θ (upper half: $\phi \le \pi/2$, $0 \le \theta \le 2\pi$), so that $-z^2 = -(2\cos\phi)^2$ and $dS = 2^2 \sin\phi \, d\phi \, d\theta$. Hence

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S -z^2 \, dS = \int_0^{2\pi} \int_0^{\pi/2} -16\cos^2\phi \sin\phi \, d\phi \, d\theta.$$

We calculate: $\iint_S \vec{F}\cdot\hat{\mathrm{n}}\,dS = -32\pi\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi = -32\pi\left[-\frac{1}{3}cos^3\phi\right]_0^{\pi/2} = -32\pi/3$.

16.7 # 29: $\vec{F} = \langle x, 2y, 3z\rangle$. We integrate separately over each of the 6 faces, noting that they are squares of side length 2 and area 4.

- front ($x=1$): $\hat{\mathrm{n}} = \hat{\imath}$, so $\vec{F}\cdot\hat{\mathrm{n}} = x = 1$, and $\iint \vec{F}\cdot d\vec{S} = \iint 1\,dS = \text{area} = 4$.
- back ($x=-1$): $\hat{\mathrm{n}} = -\hat{\imath}$, so $\vec{F}\cdot\hat{\mathrm{n}} = -x = +1$, and $\iint \vec{F}\cdot d\vec{S} = \iint 1\,dS = \text{area} = 4$.
- right ($y=1$): $\hat{\mathrm{n}} = \hat{\jmath}$, so $\vec{F}\cdot\hat{\mathrm{n}} = 2y = 2$, and $\iint \vec{F}\cdot d\vec{S} = \iint 2\,dS = 2\,\text{area} = 8$.
- left ($y=-1$): $\hat{\mathrm{n}} = -\hat{\jmath}$, so $\vec{F}\cdot\hat{\mathrm{n}} = -2y = +2$, and $\iint \vec{F}\cdot d\vec{S} = \iint 2\,dS = 2\,\text{area} = 8$.
- top ($z=1$): $\hat{\mathrm{n}} = \hat{\mathrm{k}}$, so $\vec{F}\cdot\hat{\mathrm{n}} = 3z = 3$, and $\iint \vec{F}\cdot d\vec{S} = \iint 3\,dS = 3\,\text{area} = 12$.
- bottom ($z=-1$): $\hat{\mathrm{n}} = -\hat{\mathrm{k}}$, so $\vec{F}\cdot\hat{\mathrm{n}} = -3z = +3$, and $\iint \vec{F}\cdot d\vec{S} = \iint 3\,dS = 3\,\text{area} = 12$.

Summing, the total flux is $\iint_S \vec{F}\cdot d\vec{S} = 4+4+8+8+12+12 = 48$.

16.9 # 2: $\operatorname{div}\vec{F} = 0 + 2z + 8z = 10z$, so

$$\iiint_E \operatorname{div} F\,dV = \iiint_E 10z\,dV = \int_0^{2\pi}\int_0^3\int_{r^2}^9 (10z)\,r\,dz\,dr\,d\theta.$$

Inner: $\left[5rz^2\right]_{r^2}^9 = 405r - 5r^5$. Middle: $\left[\frac{405}{2}r^2 - \frac{5}{6}r^6\right]_0^3 = 1215$. Outer: 2430π.

Next, we calculate the flux directly. Let S_1 be the surface of the paraboloid $z = x^2+y^2$ ($x^2+y^2\le 9$), oriented downwards, and S_2 the top face, i.e. the disk of radius 3 in the plane $z=9$, oriented upwards.

On S_1: the surface is $z = g(x,y) = x^2+y^2$, so $d\vec{S} = -\langle -g_x, -g_y, 1\rangle\,dx\,dy = \langle 2x, 2y, -1\rangle\,dx\,dy$.

$$\begin{aligned}\iint_{S_1}\vec{F}\cdot d\vec{S} &= \iint_{S_1}\langle y^2z^3, 2yz, 4z^2\rangle\cdot\langle 2x,2y,-1\rangle\,dx\,dy = \iint_{S_1}(2xy^2z^3 + 4y^2z - 4z^2)\,dx\,dy\\ &= \int_0^{2\pi}\int_0^3 (2r\cos\theta(r\sin\theta)^2(r^2)^3 + 4(r\sin\theta)^2r^2 - 4r^4)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^3 (2r^{10}\sin^2\theta\cos\theta + 4r^5\sin^2\theta - 4r^5)\,dr\,d\theta.\end{aligned}$$

Inner: $\left[\frac{2}{11}r^{11}\sin^2\theta\cos\theta + \frac{2}{3}r^6(\sin^2\theta - 1)\right]_0^3 = \frac{2.3^{11}}{11}\sin^2\theta\cos\theta - 2.3^5\cos^2\theta$.

Outer: $\int_0^{2\pi}\frac{2.3^{11}}{11}\sin^2\theta\cos\theta - 243(1+\cos 2\theta)\,d\theta = \left[\frac{2.3^{10}}{11}\sin^3\theta - 243(\theta + \frac{1}{2}\sin 2\theta)\right]_0^{2\pi} = -486\pi$.

On S_2: the normal vector is $\hat{\mathrm{n}} = \hat{\mathrm{k}}$, and $\vec{F}\cdot\hat{\mathrm{n}} = 4z^2 = 4.9^2 = 324$, so $\iint_{S_2}\vec{F}\cdot\hat{\mathrm{n}}\,dS = \iint_{S_2} 324\,dS = 324\,\text{area}(S_2) = 324\cdot 9\pi = 2916\pi$.

So the total flux is $-486\pi + 2916\pi = 2430\pi$, in agreement with the first calculation.

16.9 # 3: $\operatorname{div}\vec{F} = 1$, so $\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E 1\,dV = \text{volume of ball} = \frac{4}{3}\pi 4^3 = 256\pi/3$.

To calculate $\iint_S \vec{F}\cdot d\vec{S}$: the normal vector to the sphere of radius 4 is $\hat{\mathrm{n}} = \frac{1}{4}\langle x,y,z\rangle$ (pointing radially outwards), so $\vec{F}\cdot\hat{\mathrm{n}} = \frac{1}{4}\langle z,y,x\rangle\cdot\langle x,y,z\rangle = \frac{1}{2}xz + \frac{1}{4}y^2$. Parametrizing S by the spherical angles ϕ and θ, we have $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$, $z = 4\cos\phi$, and $dS = 4^2\sin\phi\,d\phi\,d\theta$. Hence

$$\begin{aligned}\iint_S \vec{F}\cdot\hat{\mathrm{n}}\,dS &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{2}(4\sin\phi\cos\theta)(4\cos\phi) + \frac{1}{4}(4\sin\phi\sin\theta)^2\right)(16\sin\phi)\,d\phi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi} 128\sin^2\phi\cos\phi\cos\theta + 64\sin^3\phi\sin^2\theta\,d\phi\,d\theta\\ &= 128\left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi\right) + 64\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right).\end{aligned}$$

Since $\int_0^{2\pi}\cos\theta\,d\theta = 0$, the first term is zero; while $\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}(\frac{1}{2} - \frac{1}{2}\cos 2\theta)\,d\theta = \pi$, and $\int_0^\pi \sin^3\phi\, d\phi = \int_0^\pi (1-\cos^2\phi)\sin\phi\, d\phi = \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^\pi = \frac{2}{3} - (-\frac{2}{3}) = \frac{4}{3}$, so $\iint_S \vec{F}\cdot \hat{\mathrm{n}}\, dS = 64\cdot\pi\cdot\frac{4}{3} = 256\pi/3$.

16.9 # 7: $\operatorname{div}\vec{F} = 3y^2 + 0 + 3z^2$, so using (rotated) cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (3y^2+3z^2)\, dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 3r^2\, r\, dx\, dr\, d\theta = 3(\int_0^{2\pi} d\theta)(\int_0^1 r^3\, dr)(\int_{-1}^2\, dx) = 3(2\pi)(\frac{1}{4})(3) = \frac{9\pi}{2}$.

16.9 # 11: $\operatorname{div}\vec{F} = 6x^2 + 3y^2 + 3y^2 = 6(x^2+y^2)$, so $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 6(x^2+y^2)\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6r^2\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 6r^3(1-r^2)\, dr\, d\theta = 2\pi\left[\frac{3}{2}r^4 - r^6\right]_0^1 = \pi$.

Problem 1. a) North of 38° N is $0 \le \phi < \phi_0$, where $\phi_0 = (90-38)\pi/180$ radians.

$$\frac{\int_0^{2\pi}\int_0^{\phi_0}\sin\phi\ d\phi d\theta}{\int_0^{2\pi}\int_0^{\pi}\sin\phi\ d\phi d\theta} = \frac{\int_0^{\phi_0}\sin\phi\ d\phi}{\int_0^{\pi}\sin\phi\ d\phi} = \frac{-\cos\phi_0 - (-1)}{2} \approx .192.$$

About 19.2 percent of the Earth's surface is north of Berkeley.

b) First take the average of ϕ:

$$\bar{\phi} = \frac{\int_0^{2\pi}\int_{\pi/2}^{\pi}\phi\sin\phi\ d\phi d\theta}{\int_0^{2\pi}\int_{\pi/2}^{\pi}\sin\phi\ d\phi d\theta} = \pi - 1$$

because $\int_{\pi/2}^{\pi}\sin\phi\ d\phi = 1$, and, by integration by parts,

$$\int_{\pi/2}^{\pi}\phi\sin\phi\, d\phi = -\int_{\pi/2}^{\pi}(-\cos\phi)\, d\phi + (-\cos\phi)\phi\Big|_{\pi/2}^{\pi} = \pi + \int_{\pi/2}^{\pi}\cos\phi\, d\phi = \pi - 1$$

Thus $\bar{\phi} = \pi - 1$ radian and the average latitude is 32.7° S. The closest large cities are Santiago, Chile (33.4° S) and Perth, Australia (31.9° S).