Math 53 Homework 11 – Solutions

16.4 # 3: a) Let C_1 be the segment from $(0,0)$ to $(1,0)$, C_2 the segment from from $(1,0)$ to $(1,2)$, and C_3 the segment from $(1,2)$ to $(0,0)$.
C_1: $x = t$, $y = 0$ $(0 \le t \le 1)$, so $dx = dt$, $dy = 0$, and $\int_{C_1} xy\,dx + x^2y^3\,dy = \int_0^1 0\,dt = 0$.
C_2: $x = 1$, $y = t$ $(0 \le t \le 2)$, so $dx = 0$, $dy = dt$, $\int_{C_2} xy\,dx + x^2y^3\,dy = \int_0^2 t^3\,dt = \left[\frac{1}{4}t^4\right]_0^2 = 4$.
$-C_3$, i.e. C_3 backwards from $(0,0)$ to $(1,2)$: $x = t$, $y = 2t$, $0 \le t \le 1$. So $dx = dt$ and $dy = 2\,dt$. $\int_{-C_3} xy\,dx + x^2y^3\,dy = \int_0^1 (2t^2 + 16t^5)\,dt = \left[\frac{2}{3}t^3 + \frac{8}{3}t^6\right]_0^1 = \frac{10}{3}$. So $\int_{C_3} = -10/3$.
(Note: without switching the orientation of C_3, one could parametrize it as $x = 1 - t$, $y = 2 - 2t$, $0 \le t \le 1$; this gives the same answer.)
Finally: $\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + 4 - \frac{10}{3} = \frac{2}{3}$.
b) by Green's theorem, $\oint_C xy\,dx + x^2y^3\,dy = \iint_R \left(\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right) dA$, where R is the region enclosed by C. Thus $\oint_C xy\,dx + x^2y^3\,dy = \iint_R 2xy^3 - x\,dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$.
Inner: $\int_0^{2x} (2xy^3 - x)\,dy = \left[\frac{1}{2}xy^4 - xy\right]_0^{2x} = 8x^5 - 2x^2$.
Outer: $\int_0^1 8x^5 - 2x^2\,dx = \left[\frac{4}{3}x^6 - \frac{2}{3}x^3\right]_0^1 = \frac{2}{3}$.

16.4 # 9: $\int_C y^3\,dx - x^3\,dy = \iint_R \left(\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right) dA = \iint_R (-3x^2 - 3y^2)\,dA$, where $R = \{x^2 + y^2 \le 4\}$. So $\iint_R (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 -3r^2\,r\,dr\,d\theta = 2\pi \cdot \left[-\frac{3}{4}r^4\right]_0^2 = -24\pi$.

16.4 # 13: C is clockwise and encloses the disk R of radius 2 centered at $(3,-4)$.
Therefore $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = -\oint_{-C} \vec{F} \cdot d\vec{r} = -\iint_R \left(\frac{\partial}{\partial x}(x \sin y) - \frac{\partial}{\partial y}(y - \cos y)\right) dA =$

$$= -\iint_R -1\,dA = \iint_R 1\,dA = \text{area}(R) = \pi 2^2 = 4\pi.$$

16.4 # 19: Let C_1 be the arch of cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$ (so C_2 is given by $x = 2\pi - t$, $y = 0$ for $0 \le t \le 2\pi$). Then $C = C_1 + C_2$ is traversed clockwise, so $-C$ is oriented positively and encloses the area under one arch of the cycloid. By formula (5) in §16.4,
$A = \oint_{-C} -y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt + \int_0^{2\pi} 0\,(-dt) =$
$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt = \int_0^{2\pi} (1 - 2\cos t + \frac{1 + \cos t}{2})\,dt = \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$.

16.4 # 25: Let D be the region enclosed by C. Then by Green's theorem,

$$-\frac{\rho}{3}\oint_C y^3\,dx = -\frac{\rho}{3}\iint_D -\frac{\partial}{\partial y}(y^3)\,dA = \frac{\rho}{3}\iint_D 3y^2\,dA = \iint_A y^2\,\rho\,dA = I_x.$$

Similarly, $\displaystyle\frac{\rho}{3}\oint_C x^3\,dy = \frac{\rho}{3}\iint_D \frac{\partial}{\partial x}(x^3)\,dA = \frac{\rho}{3}\iint_D 3x^2\,dA = \iint_A x^2\,\rho\,dA = I_y$.

16.5 # 9: If we write $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$, then $P = 0$, while Q is independent of x (and z), and decreases as y increases, so $\partial Q/\partial y < 0$.
a) $\operatorname{div} \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = 0 + \dfrac{\partial Q}{\partial y} < 0$.
b) since we are considering a 2D vector field, the x and y components of the curl vanish, and $\operatorname{curl} \vec{F} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\hat{k} = (0 - 0)\hat{k} = 0$.

(This illustrates the interpretation of divergence for velocity fields: the field $\vec{F}$ corresponds to a flow that "compresses" areas, hence div< 0. On the other hand, there is no "spinning" motion, hence curl is zero).

16.5 # 11: If we write $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$, then $Q = 0$, while P is independent of x (and z), and increases as y increases, so $\partial P/\partial y > 0$.

a) $\operatorname{div}\vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = 0 + 0 = 0$.

b) since we are considering a 2D vector field, the x and y components of the curl vanish, and $\operatorname{curl}\vec{F} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\hat{k} = -\dfrac{\partial P}{\partial y}\hat{k}$ points in the negative z direction.

(This illustrates the interpretation of curl for velocity fields in terms of rotation, see also 16.5 # 37. The flow described by $\vec{F}$ is neither compressing nor expanding, hence div $= 0$; but the "shearing" motion causes a particle placed in this velocity field to spin clockwise).

16.5 # 13: $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2y)\hat{\imath} - (3y^2z^2 - 3y^2z^2)\hat{\jmath} + (2yz^3 - 2yz^3)\hat{k} = 0$. Since $\vec{F}$ is defined everywhere and $\operatorname{curl}\vec{F} = 0$, $\vec{F}$ is conservative.

$f(x, y, z)$ is a potential function if it satisfies: $f_x = y^2z^3$, $f_y = 2xyz^3$, $f_z = 3xy^2z^2$.

Integrating $f_x = y^2z^3$, we get that $f(x, y, z) = xy^2z^3 + g(y, z)$ for some function $g(y, z)$. Thus $f_y = 2xyz^3 + g_y = 2xyz^3$, which implies that $g_y = 0$, hence $g(y, z) = h(z)$ depends on z only, and $f(x, y, z) = xy^2z^3 + h(z)$. Finally, $f_z = 3xy^2z^2 + h'(z) = 3xy^2z^2$, so $h'(z) = 0$, so $h(z) = c$ for some constant c. Hence $f(x, y, z) = xy^2z^3 + c$.

16.5 # 25: if $\vec{F} = \langle P, Q, R\rangle$ then $\operatorname{div}(f\vec{F}) = \dfrac{\partial(fP)}{\partial x} + \dfrac{\partial(fQ)}{\partial y} + \dfrac{\partial(fR)}{\partial z} =$

$= (fP_x + f_xP) + (fQ_y + f_yQ) + (fR_z + f_zR)$

$= f(P_x + Q_y + R_z) + \langle P, Q, R\rangle \cdot \langle f_x, f_y, f_z\rangle = f\operatorname{div}(\vec{F}) + \vec{F} \cdot \nabla f$.

16.5 # 33: Using Green's theorem in normal form, and the result of 16.5 # 25,

$$\oint_C f(\nabla g) \cdot \hat{n}\, ds = \iint_D \operatorname{div}(f\nabla g)\, dA = \iint_D (f\operatorname{div}(\nabla g) + \nabla g \cdot \nabla f)\, dA.$$

However, $\operatorname{div}(\nabla g) = \nabla^2 g$ by definition, see p. 1107 (7th ed: p. 1095). Hence

$$\iint_D f\nabla^2 g\, dA = \oint_C f(\nabla g) \cdot \hat{n}\, ds - \iint_D \nabla g \cdot \nabla f\, dA.$$

16.5 # 36: Green's first identity (the result of 16.5 # 33) for $g = f$ reads:

$$\iint_D f\nabla^2 f\, dA = \oint_C f\nabla f \cdot \hat{n}\, ds - \iint_D |\nabla f|^2\, dA.$$

If we assume that f is harmonic in D, i.e. $\nabla^2 f = 0$, then the left hand side is zero. If moreover we assume that f is zero at every point of C, then $\oint_C f\nabla f \cdot \hat{n}\, ds = \oint_C 0\, ds = 0$. Therefore, we conclude that

$$\iint_D |\nabla f|^2\, dA = 0.$$

(Since we assume continuity of the partial derivatives of f, this implies that $|\nabla f|^2 = 0$ everywhere in D, hence $\nabla f = 0$, hence f is constant in D; however, because $f = 0$ at the boundary of D, this implies that f is zero everywhere in D; an important result about harmonic functions.)

16.5 # 37: a) We know that the speed v at the given point P (at distance d from the z axis) is related to the angular speed ω by $\omega = v/d$, i.e. $v = \omega d$. However, $d = |\vec{r}|\sin\theta$ where $\vec{r} = \overrightarrow{OP}$ and θ is as in the figure; so $v = \omega|\vec{r}|\sin\theta = |\vec{w} \times \vec{r}|$. Moreover $\vec{v}$ is perpendicular to both $\vec{w}$ and $\vec{r}$, and the right-hand rule holds (cf. figure), so $\vec{v} = \vec{w} \times \vec{r}$.

b) From (a), $\vec{v} = \vec{w} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \langle -\omega y, \omega x, 0 \rangle$.

c) $\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$= (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y) \right) \hat{k} = (\omega - (-\omega))\hat{k} = 2\omega\hat{k} = 2\vec{w}.$$

Problem 1: a) Let C be the unit circle $x^2 + y^2 = 1$, oriented counterclockwise, and let R be the unit disk (the region enclosed by C). By Green's theorem,

$$\int_C x^{2n-1} dy = \iint_R \frac{\partial}{\partial x}(x^{2n-1})\, dA = \iint_R (2n-1)x^{2n-2}\, dA. \qquad (1)$$

The integral on the left side of (1) is $\displaystyle\int_C x^{2n-1} dy = \int_0^{2\pi} \cos^{2n-1}\theta \, \cos\theta \, d\theta = I_n$.

The integral on the right side of (1) is

$$\iint_R (2n-1)x^{2n-2}\, dA = (2n-1)\int_0^{2\pi}\int_0^1 (r\cos\theta)^{2n-2}\, r dr d\theta$$

$$= (2n-1)\int_0^{2\pi} \cos^{2n-2}\theta \left[\frac{r^{2n}}{2n}\right]_0^1 d\theta = \frac{2n-1}{2n}\int_0^{2\pi} \cos^{2n-2}\theta d\theta = \frac{2n-1}{2n} I_{n-1}.$$

Thus, $I_n = \frac{2n-1}{2n} I_{n-1}$.

b) $I_0 = \int_0^{2\pi} d\theta = 2\pi$. Therefore, by the result of (a) (with $n = 1$), $I_1 = \frac{1}{2} I_0 = 2\pi \cdot \frac{1}{2}$; similarly, $I_2 = \frac{3}{4} I_1 = 2\pi \cdot \frac{1}{2} \cdot \frac{3}{4}$, and $I_3 = \frac{5}{6} I_2 = 2\pi \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}$. More generally,

$$I_n = \frac{2n-1}{2n} I_{n-1} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} I_{n-2} = \cdots = 2\pi \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot 2n-1}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}.$$

Problem 2:

a) The vector $\vec{F}(x,y) = x^2\hat{\imath} + xy\hat{\jmath} = x(x\hat{\imath} + y\hat{\jmath})$ is parallel to the position vector $x\hat{\imath} + y\hat{\jmath}$ of the point (x, y), so it always points in the radial direction: straight away from the origin if $x > 0$, straight towards the origin if $x < 0$, and it vanishes if $x = 0$.

In particular, $\vec{F}$ points outwards of the unit circle C for $x > 0$, and inwards for $x < 0$. So the normal component $\vec{F} \cdot \hat{n}$, which is the integrand of the line integral for flux, is positive for $x > 0$ (the right half of the unit circle), and negative for $x < 0$ (the left half).

b) The normal vector to C at (x, y) is $\hat{n} = x\hat{\imath} + y\hat{\jmath}$. Therefore $\vec{F} \cdot \hat{n} = \langle x^2, xy \rangle \cdot \langle x, y \rangle = x^3 + xy^2 = x(x^2 + y^2) = x$. (Recall $x^2 + y^2 = 1$ on the unit circle.) This is indeed positive for $x > 0$ and negative for $x < 0$.

We parametrize the unit circle by the polar angle θ: so $x = \cos\theta$, $y = \sin\theta$, and the length element is $ds = d\theta$. Therefore $\int_C \vec{F} \cdot \hat{n}\, ds = \int_C x\, ds = \int_0^{2\pi} \cos\theta\, d\theta = 0$.

We get zero because, by symmetry about the y-axis, the (negative) flux through the left half-circle exactly compensates the (positive) flux through the right half-circle.

c) $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) = 3x$. So by Green's theorem, we have

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R 3x\, dA = 3\iint_R x\, dA = 0$$

where R is the unit disk, and $\iint_R x\,dA$ is seen to be zero either by symmetry or by computation in polar coordinates

$$\iint_R x\,dA = \int_0^{2\pi}\int_0^1 r\cos\theta\, r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{3}\cos\theta\,d\theta = 0.$$

Problem 3:

a) For the circle C_a of radius a centered at the origin: unit normal: $\hat{\text{n}} = \langle x,y\rangle/a$, and $\vec{F}\cdot\hat{\text{n}} = \langle x/a^2, y/a^2\rangle\cdot\langle x,y\rangle/a = (x^2+y^2)/a^3 = 1/a$. So

$$\int_{C_a}\vec{F}\cdot\hat{\text{n}}\,ds = \int_{C_a}\frac{1}{a}\,ds = \frac{1}{a}(2\pi a) = 2\pi.$$

b) $\operatorname{div}\vec{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \dfrac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = 0.$

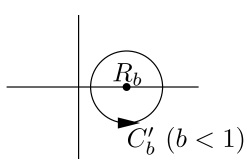
So, for $b<1$, the flux out of the circle C'_b around $(1,0)$ of radius b is zero, because $\int_{C'_b}\vec{F}\cdot\hat{\text{n}}\,ds = \iint_{R_b}\operatorname{div}\vec{F}\,dA = 0$, where R_b is the disk enclosed by C'_b.

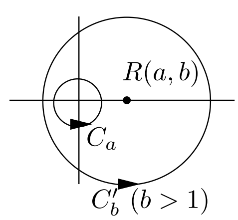
Next, consider $b>1$. The previous argument does not work because R_b contains the origin, where $\vec{F}$ is not defined. To avoid the origin, we consider the region $R(a,b) = R_b - D_a$, the disk R_b with the disk D_a of radius a around the origin removed. Let C_a be the circle of radius a around the origin, as in part (a), oriented counterclockwise. For a sufficiently small, C_a is inside of R_b and the oriented boundary of $R(a,b)$ is $C'_b - C_a$ (see figure below). By Green's theorem,

$$\int_{C'_b}\vec{F}\cdot\hat{\text{n}}\,ds - \int_{C_a}\vec{F}\cdot\hat{\text{n}}\,ds = \iint_{R(a,b)}\operatorname{div}\vec{F}\,dA = 0$$

(One can apply Green's theorem because $\vec{F}$ and $\operatorname{div}\vec{F}$ are defined in $R(a,b)$.) Moreover, by part (a) the flux through C_a is equal to 2π. Hence, for $b>1$,

$$\int_{C'_b}\vec{F}\cdot\hat{\text{n}}\,ds = \int_{C_a}\vec{F}\cdot\hat{\text{n}}\,ds = 2\pi.$$





16.6 #13: The parametric equations for the surface are: $x = u\cos v$, $y = u\sin v$, $z = v$. We look at the grid lines (the curves obtained by fixing one of u and v and varying the other).

If we keep v constant and vary u, then x and y move along a straight line in the horizontal plane $z = v$, intersecting the z-axis at $(0,0,v)$ (for $u=0$) and directed along the vector $\langle\cos v, \sin v, 0\rangle$ (a horizontal vector whose direction rotates with the height $z=v$).

If we fix u and vary v, then the projection to the xy-plane is a circle ($x = u\cos v$, $y = u\sin v$, so $x^2+y^2 = u^2$) while $z=v$ increases at a constant rate, so we obtain a helix.

Thus we obtain the surface shown on graph IV (called a *helicoid*).

16.6 # 23: The cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$, and we want the portion of the sphere above this. Using x and y as parameters, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$, where $x^2 + y^2 \leq 2$.

Alternatively, in cylindrical coordinates, using r and θ as parameters, we can parametrize the surface by $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{4 - r^2}$, where $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$.

Or, using spherical coordinates, where the sphere is $\rho = 2$ and the cone is $\phi = \pi/4$: using ϕ and θ as parameters, the parametrization becomes $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$, where $0 \leq \phi \leq \pi/4$ and $0 \leq \theta \leq 2\pi$.

16.6 # 24: $x^2 + z^2 = 9$ is a cylinder of radius 3 centered on the y-axis; its projection to the xz-plane is the circle of radius 3, easiest to parametrize by $x = 3\cos\theta$, $z = 3\sin\theta$. The half where $z \geq 0$ (above the xy-plane) corresponds to $0 \leq \theta \leq \pi$. Meanwhile, y varies between -4 and 4. Hence: $x = 3\cos\theta$, $y = y$, $z = 3\sin\theta$, $-4 \leq y \leq 4$, $0 \leq \theta \leq \pi$.

(Or, using rectangular coordinates, since we are only considering the upper half of the cylinder where $z = \sqrt{9 - x^2}$: $x = x$, $y = y$, $z = \sqrt{9 - x^2}$, $-3 \leq x \leq 3$, $-4 \leq y \leq 4$.)

16.6 # 44: $z = f(x, y) = 4 - 2x^2 + y$ over the triangle D with vertices $(0,0)$, $(1,0)$, $(1,1)$ (i.e.: $0 \leq y \leq x$, $0 \leq x \leq 1$):
$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_D \sqrt{1 + (-4x)^2 + 1^2}\, dA = \int_0^1 \int_0^x \sqrt{2 + 16x^2}\, dy\, dx$.
Inner: $\left[y\sqrt{2 + 16x^2}\right]_0^x = x\sqrt{2 + 16x^2}$.
Outer: $\int_0^1 x\sqrt{2 + 16x^2}\, dx = \left[\frac{1}{48}(2 + 16x^2)^{3/2}\right]_0^1 = \frac{1}{48}(18^{3/2} - 2^{3/2}) = \frac{1}{48}(54\sqrt{2} - 2\sqrt{2}) = \frac{13\sqrt{2}}{12}$.

16.6 # 45: $z = f(x, y) = xy$ over the unit disk $D : x^2 + y^2 \leq 1$. Since $f_x = y$ and $f_y = x$,
$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_D \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\; r\, dr\, d\theta$.
So $A(S) = 2\pi \int_0^1 \sqrt{r^2 + 1}\; r\, dr = 2\pi \left[\frac{1}{3}(r^2 + 1)^{3/2}\right]_0^1 = \frac{2\pi}{3}(2\sqrt{2} - 1)$.