Math 53 Homework 11

Due Wednesday 11/8/17 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 10/30: Green's theorem

- **Read:** section 16.4.
- **Work:** 16.4: (1), (2), 3, (4), (7), 9, (11), 13, (17), 19, (21), 25, (26).
  Problem 1 below.

Wednesday 11/1: Curl and divergence; vector forms of Green's theorem

- **Read:** section 16.5.
- **Work:** 16.5: (1), (7), 9, 11, (12), 13, (16), (21), 25, (26), (32), 33, (34), 36*, 37.
  Problems 2 and 3 below.

* Remark: 16.5 # 36 shows that, if a harmonic function f on a domain D is zero everywhere at its boundary curve C, then $f = 0$ everywhere in D. A consequence is that, if f_1 and f_2 are two harmonic functions in D such that $f_1 = f_2$ at every point of C, then $f_1 = f_2$ everywhere in D (by applying the previous result to the harmonic function $f_1 - f_2$). This uniqueness property has important applications in mathematical analysis.

Friday 11/3: Surface area

- **Read:** section 16.6.
- **Work:** 16.6: (3), 13, (18), 23, 24[1], (25), (32), (39), 44[1], 45, (47).

Problem 1. This problem shows how the value of the integral $I_n = \int_0^{2\pi} \cos^{2n}\theta \, d\theta$ can be determined for all n using Green's theorem (the conventional method is via integration by parts).

a) Prove that $I_n = \dfrac{2n-1}{2n} I_{n-1}$ by writing Green's theorem for the line integral $\oint_C x^{2n-1}\, dy$, where C is the unit circle $x^2 + y^2 = 1$ counterclockwise, and by evaluating each of the integrals separately until it looks like either I_n or I_{n-1}.

b) What is the value of I_0? Using the result of (a), find expressions for I_1, I_2, I_3 (don't simplify fractions or calculate products), then give a general formula for I_n.

[1] **7th ed:** do the 8th ed problems: **# 24:** the part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$. **# 44:** the part of the surface $z = 4 - 2x^2 + y$ that lies above the triangle with vertices (0,0), (1,0), and (1,1).

Problem 2.

a) Let C be the unit circle, oriented counterclockwise, and consider the vector field $\vec{F} = x^2\,\hat{\imath} + xy\,\hat{\jmath}$. Describe geometrically and/or sketch the vector field $\vec{F}$ (see also 16.2 # 32 (b) assigned on HW 10). Which portions of C contribute positively to the flux $\int_C \vec{F} \cdot \hat{n}\, ds$? Which portions contribute negatively?

b) Find the flux of $\vec{F}$ through C by directly evaluating the line integral $\int_C \vec{F} \cdot \hat{n}\, ds$. Explain your answer using (a).

c) Find the flux of $\vec{F}$ through C by using the second vector form ("normal form") of Green's theorem.

Problem 3.

Find the flux of the vector field $\vec{F} = \dfrac{x\hat{\imath} + y\hat{\jmath}}{x^2 + y^2}$ outwards through:

a) the circle of radius a centered at the origin; do this by calculating the line integral $\int \vec{F} \cdot \hat{n}\, ds$ directly.

b) the circle centered at $(1, 0)$ of radius b, for $b \neq 1$. Consider the cases $b > 1$ and $b < 1$ separately, and use Green's theorem (carefully!). Explain your answers with diagrams.