Math 53 Homework 10 – Solutions

16.1 # 11: $\vec{F}(x,y) = \langle x, -y\rangle$ corresponds to graph IV. In the first quadrant all the vectors have positive x-components and negative y-components, in the second quadrant they have negative x- and y-components, etc. Also, the vectors get shorter as we approach the origin.

16.1 # 13: $\vec{F}(x,y) = \langle y, y+2\rangle$ corresponds to graph I since the vectors are independent of x (the vectors along horizontal lines are all identical) and, as y increases (towards the top of the diagram), both the x- and the y-components get larger.

16.1 # 18: $\vec{F}(x,y,z) = \langle x,y,z\rangle$ corresponds to graph II: each vector $\vec{F}(x,y,z)$ has the same length and direction as the position vector of the point (x,y,z), and therefore the vectors all point directly away from the origin.

16.1 # 23: $\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle =$
$= \langle x,y,z\rangle/\sqrt{x^2+y^2+z^2}$. This vector has the same direction as $\langle x,y,z\rangle$ (= position vector of the point (x,y,z)), so points directly away from the origin ("radially"); while its magnitude is $(x^2+y^2+z^2)^{-1/2}|\langle x,y,z\rangle| = 1$. So $\nabla f(x,y,z)$ is the unit vector in the direction of $\langle x,y,z\rangle$ (pointing directly away from the origin).

16.1 # 31: $f(x,y) = (x+y)^2 \Rightarrow \nabla f = \langle 2(x+y), 2(x+y)\rangle = 2(x+y)(\hat{\imath}+\hat{\jmath})$. So all the vectors are parallel to $\hat{\imath}+\hat{\jmath}$; they vanish on the line $x+y=0$ (or $y=-x$), and their magnitude increases with the distance to that line. This corresponds to plot II.

16.2 # 1: $x = t^2$ and $y = 2t$, so $\int_C y\,ds = \int_0^3 2t\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt = \int_0^3 2t\sqrt{(2t)^2+2^2}\,dt$
$= \int_0^3 4t\sqrt{t^2+1}\,dt = \left[\frac{4}{3}(t^2+1)^{3/2}\right]_0^3 = \frac{4}{3}(10^{3/2}-1).$

16.2 # 3: Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. So
$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2+(4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^6\cos t\sin^4 t\,dt =$
$= 4^6\left[\frac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \frac{2}{5}4^6 = \frac{2^{13}}{5} = \frac{8192}{5}.$

16.2 # 17: (a) Along the line $x = -3$, the vectors of $\vec{F}$ have positive y-components; since the path goes upward, the integrand $\vec{F}\cdot\hat{T}$ is always positive. So $\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot\hat{T}\,ds > 0$.
(b) Along the circle C_2, the field vectors are all pointing in the clockwise direction, i.e., opposite the direction of the path. So $\vec{F}\cdot\hat{T} < 0$, and therefore $\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot\hat{T}\,ds < 0$.

16.2 # 20: $x = t^2$, $y = t^3$, $z = -2t$, so $dx = 2t\,dt$, $dy = 3t^2\,dt$, $dz = -2\,dt$. Hence

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_C (x+y^2)\,dx + xz\,dy + (y+z)\,dz \\
&= \int_0^2 (t^2+t^6)(2t)\,dt + (-2t^3)(3t^2)\,dt + (t^3-2t)(-2)\,dt \\
&= \int_0^2 (4t - 6t^5 + 2t^7)\,dt = \left[2t^2 - t^6 + \frac{1}{4}t^8\right]_0^2 = 2^3 - 2^6 + \frac{1}{4}2^8 = 8.
\end{aligned}$$

16.2 # 32: (a) We parametrize the circle C by: $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. So $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$, and

$$\int_C \vec{F}\cdot d\vec{r} = \int_C x^2\,dx + xy\,dy = \int_0^{2\pi} (2\cos t)^2\,(-2\sin t)\,dt + (4\cos t\sin t)\,(2\cos t)\,dt$$
$$= \int_0^{2\pi} (-8\cos^2 t\sin t + 8\cos^2 t\sin t)\,dt = \int_0^{2\pi} 0\,dt = 0.$$

(b) The vector $\vec{F}(x,y) = x^2\hat{\imath} + xy\hat{\jmath} = x(x\hat{\imath} + y\hat{\jmath})$ is parallel to the position vector $x\hat{\imath} + y\hat{\jmath}$ of the point (x,y), so it always points in the radial direction (straight away from the origin if $x > 0$, towards the origin if $x < 0$). (This can be seen on a plot). So, at every point of the circle C, the vector $\vec{F}(x,y)$ is perpendicular to the circle, hence the field does no work on the moving particle. In other words, $\vec{F}\cdot\hat{T} = 0$ at any point along C, and so $\int_C \vec{F}\cdot d\vec{r} = 0$.

16.2 # 42: The line segment from (2,0,0) to (2,1,5) has parametric equations $x = 2$, $y = t$, $z = 5t$ for $0 \le t \le 1$; also, $\vec{F} = K\vec{r}/|\vec{r}|^3 = K(x^2+y^2+z^2)^{-3/2}\langle x,y,z\rangle$. So:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C K\frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} = K\int_0^1 \frac{(2)(0)\,dt + (t)(1)\,dt + (5t)(5)\,dt}{(4+t^2+(5t)^2)^{3/2}} =$$
$$K\int_0^1 \frac{26t\,dt}{(4+26t^2)^{3/2}} = K\left[\frac{-1}{(4+26t^2)^{1/2}}\right]_0^1 = K\left(\frac{1}{2} - \frac{1}{\sqrt{30}}\right).$$

Problem 1: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2y + \frac{1}{3}y^3)\,dx = \int_{x_1}^{x_2} \left(x^2 f(x) + \frac{1}{3}f(x)^3\right)dx,$ and

$$\iint_R (x^2+y^2)\,dA = \int_{x_1}^{x_2}\int_0^{f(x)} (x^2+y^2)\,dy\,dx = \int_{x_1}^{x_2}\left[x^2y + \frac{1}{3}y^3\right]_0^{f(x)} dx$$
$$= \int_{x_1}^{x_2}\left(x^2 f(x) + \frac{1}{3}f(x)^3\right)dx.$$

These two integrals are therefore equal.

16.3 # 7: $\frac{\partial}{\partial y}(ye^x + \sin y) = e^x + \cos y = \frac{\partial}{\partial x}(e^x + x\cos y)$, and $\vec{F}$ is defined in the entire plane (simply connected), so $\vec{F}$ is conservative. So there is a function f such that $\nabla f = \vec{F}$, i.e. $f_x = ye^x + \sin y$ and $f_y = e^x + x\cos y$.
Integrating with respect to x, $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ for some function $g(y)$; differentiating both sides of this equation with respect to y gives $f_y = e^x + x\cos y + g'(y)$. Thus we should have $g'(y) = 0$, and $g(y) = c$ is a constant. Hence $f(x,y) = ye^x + x\sin y + c$.

16.3 # 15: a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$, and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$, so $g_y(y,z) = 0$, and (integrating with respect to y) $g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$, and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z$, and hence $h(z) = z^2 + c$. Thus (taking $c = 0$), one potential function is $f(x,y,z) = xyz + z^2$.
b) By the fundamental theorem, $\int_C \vec{F}\cdot d\vec{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

16.3 # 19: $\frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y} = \frac{\partial}{\partial x}(2y - x^2e^{-y})$, and these quantities are continuously differentiable in the whole plane (simply connected). Thus the line integral $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$ is path-independent and can be computed using the fundamental theorem for line integrals. We first determine a potential function f such that $\nabla f = \langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$.

The equation $f_x = 2xe^{-y}$ implies that $f(x,y) = x^2e^{-y} + g(y)$ for some function $g(y)$. Hence $f_y = -x^2e^{-y} + g'(y) = 2y - x^2e^{-y}$, and so $g'(y) = 2y$, i.e. $g(y) = y^2 + c$. Thus $f(x,y) = x^2e^{-y} + y^2$ is a potential function. Therefore, given any path C from (1,0) to (2,1), the line integral is equal to $f(2,1) - f(1,0) = (4e^{-1} + 1) - 1 = 4e^{-1}$.

16.3 # 25: We know that if the vector field (call it $\vec{F}$) is conservative, then around any closed path C we must have $\int_C \vec{F} \cdot d\vec{r} = 0$. However, take C to be a circle centered at the origin, oriented counterclockwise. Then all of the field vectors along C point forward (have a positive tangential component), so $\int_C \vec{F} \cdot d\vec{r} > 0$. Therefore the field is not conservative.

Problem 2. a) For $\theta(x,y) = \tan^{-1}(y/x)$:

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = -\frac{y}{x^2+y^2}, \quad \text{and} \quad \frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}; \quad \text{so} \quad \nabla\theta = \vec{F}.$$

b) Because $\theta(x,y) = \tan^{-1}(y/x)$ is well-defined in the right half-plane ($x > 0$) and $\vec{F} = \nabla\theta$, the fundamental theorem for line integrals implies $\int_C \vec{F} \cdot d\vec{r} = \theta(x_2,y_2) - \theta(x_1,y_1) = \theta_2 - \theta_1$.

c) $$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^\pi \frac{(-\sin\theta)(-\sin\theta) + \cos\theta\cos\theta}{\cos^2\theta + \sin^2\theta}\,d\theta = \int_0^\pi d\theta = \pi.$$

Similarly, $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{-\pi} d\theta = -\int_{-\pi}^0 d\theta = -\pi.$

(Or geometrically: length(C_1) = length$(C_2) = \pi$, $\vec{F} \cdot \hat{T} = 1$ on C_1; $\vec{F} \cdot \hat{T} = -1$ on C_2)

d) $\vec{F}$ is defined everywhere except at the origin, but is not conservative over its entire domain of definition. Indeed, the two line integrals computed in (c) both run from (1,0) to $(-1,0)$ but they are not equal, so path-independence fails. On the other hand, $\vec{F}$ is conservative over the half-plane $x > 0$, where $\vec{F} = \nabla\theta$ and the fundamental theorem of calculus gives a formula for the line integral involving only the values of θ at the end points (as seen in (b)).

Note: even though $P_y = Q_x$ at every point where $\vec{F}$ is defined, we cannot conclude that $\vec{F}$ is the gradient of a well-defined potential function everywhere! This would only be true if $\vec{F}$ were defined over the entire plane (or more generally, a *simply connected region*). In fact, we can find a potential function for $\vec{F}$ over smaller regions such as the right half-plane $x > 0$ (namely, the polar angle θ). However, if we consider the entire plane with just the origin removed, the polar angle coordinate θ is not well-defined as a single-valued differentiable function: its value "jumps" by 2π as we go around the origin. This is what causes conservativeness to fail.

Problem 3: a) If $\vec{F} = r^n(x\,\hat{\imath} + y\,\hat{\jmath}) = P\hat{\imath} + Q\hat{\jmath}$ then

$$Q_x = \frac{\partial(yr^n)}{\partial x} = nyr^{n-1}\frac{x}{r}, \text{ while } P_y = \frac{\partial(xr^n)}{\partial y} = nxr^{n-1}\frac{y}{r}. \quad \text{So } P_y = Q_x.$$

(Recall $r = \sqrt{x^2+y^2}$ gives $r_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$, and similarly $r_y = \dfrac{y}{r}$.)

b) If $g = g(r)$, then $g_x = g'(r)\,\dfrac{x}{r}$ and $g_y = g'(r)\,\dfrac{y}{r}$ (by the chain rule).

So $\nabla g = \dfrac{g'(r)}{r}(x\,\hat{\imath} + y\,\hat{\jmath})$. We must find g such that $g'(r)/r = r^n$, i.e. $g'(r) = r^{n+1}$.

Two cases: $n \neq -2$: $g(r) = \dfrac{1}{n+2}\,r^{n+2}$. $\qquad n = -2$: $g(r) = \ln(r)$.