Math 53 Practice Midterm 1 A – Solutions

Problem 1.

Area: $\displaystyle\int_0^{\pi/2} \frac{1}{2}(\sqrt{\sin 2\theta})^2\,d\theta = \int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\,d\theta = -\frac{1}{4}\cos(2\theta)\Big|_0^{\pi/2} = -\frac{1}{4}((-1)-1) = \frac{1}{2}.$

Problem 2.

a) $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + 2\hat{k}.$ Area $= \frac{1}{2}|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| = \frac{1}{2}\sqrt{6}.$

b) Normal vector: $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \hat{\imath} + \hat{\jmath} + 2\hat{k}.$ Equation: $x + y + 2z = 3.$

c) Parametric equations for the line: $x = -1 + t$, $y = t$, $z = t$.

Substituting: $-1 + 4t = 3$, $t = 1$, intersection point $(0, 1, 1)$.

Problem 3.

a) $\dfrac{d}{dt}(\vec{r} \cdot \vec{r}) = \vec{v} \cdot \vec{r} + \vec{r} \cdot \vec{v} = 2\vec{r} \cdot \vec{v}.$

b) Assume $|\vec{r}|$ is constant: then $\dfrac{d}{dt}(\vec{r} \cdot \vec{r}) = 2\vec{r} \cdot \vec{v} = 0$, i.e. $\vec{r} \perp \vec{v}$.

c) $\vec{r} \cdot \vec{v} = 0$, so $\dfrac{d}{dt}(\vec{r} \cdot \vec{v}) = \vec{v} \cdot \vec{v} + \vec{r} \cdot \vec{a} = 0$. Therefore $\vec{r} \cdot \vec{a} = -|\vec{v}|^2$.

Problem 4.

a) By measuring, $\Delta h = 100$ for $\Delta s \simeq 500$, so $D_{\hat{u}} h \simeq \dfrac{\Delta h}{\Delta s} \simeq 0.2.$

b) Q is the northernmost point on the curve $h = 2200$; the vertical distance between consecutive level curves is about $1/3$ of the given length unit, so $\dfrac{\partial h}{\partial y} \simeq \dfrac{\Delta h}{\Delta y} \simeq \dfrac{-100}{1000/3} \simeq -0.3.$

Problem 5.

a) $\nabla f = (y - 4x^3)\,\hat{\imath} + x\,\hat{\jmath}$; at P, $\nabla f = \langle -3, 1 \rangle$.

b) $\Delta w \simeq -3\,\Delta x + \Delta y$.

Problem 6.

$f(x,y,z) = x^3 y + z^2 = 3$: the normal vector is $\nabla f = \langle 3x^2 y, x^3, 2z \rangle = \langle 3, -1, 4 \rangle$. The tangent plane is $3x - y + 4z = 4$.

Problem 7.

$\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y\,f_u + \dfrac{1}{y}\,f_v.$ $\dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x\,f_u - \dfrac{x}{y^2}\,f_v.$ (chain rule)

Problem 8.

a) The volume is $xyz = xy(1-x^2-y^2) = xy-x^3y-xy^3$. Critical points: $f_x = y-3x^2y-y^3 = 0$, $f_y = x - x^3 - 3xy^2 = 0$.

b) Assuming $x > 0$ and $y > 0$, the equations can be rewritten as $1-3x^2-y^2 = 0$, $1-x^2-3y^2 = 0$. Solution: $x^2 = y^2 = 1/4$, i.e. $(x, y) = (1/2, 1/2)$.

At this point, $f_{xx} = -6xy = -3/2$, $f_{yy} = -6xy = -3/2$, $f_{xy} = 1 - 3x^2 - 3y^2 = -1/2$. So $f_{xx}f_{yy} - f_{xy}^2 > 0$, and $f_{xx} < 0$, it is a local maximum.

c) The maximum of f lies either at $(1/2, 1/2)$, or on the boundary of the domain or at infinity. Since $f(x, y) = xy(1 - x^2 - y^2)$, $f = 0$ when either $x \to 0$ or $y \to 0$, and $f \to -\infty$ when $x \to \infty$ or $y \to \infty$ (since $x^2 + y^2 \to \infty$). So the maximum is at $(x, y) = (\frac{1}{2}, \frac{1}{2})$, where $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$.

Problem 9.

a) $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + y^2 + z = 1$: one must solve the Lagrange multiplier equation $\nabla f = \lambda \nabla g$, i.e. $yz = 2\lambda x$, $xz = 2\lambda y$, $xy = \lambda$, and the constraint equation $x^2 + y^2 + z = 1$.

b) The last equation gives $\lambda = xy$; substituting into the first two equations, we get $yz = 2x^2y$ and $xz = 2xy^2$, which simplify to $z = 2x^2$ and $z = 2y^2$. In particular, $y^2 = x^2$, and since $x > 0$ and $y > 0$ we get $y = x$. Substituting into the constraint equation, we get $4x^2 = 1$, so $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{1}{2}$.