Mirror symmetry for hypersurfaces ⊆ toric varieties

reference = AAK

Motivating question) How do we construct a mirror to an object H that does not admit a Lagr torus fibration?

Answer) Embed H as a hypersurface in a toric variety V, do SYZ on $Bl_{H\times 0} V\times\mathbb{C}$.

works well for finding tori:
toric = { $\exists$ dense subset $\cong(\mathbb{C}^*)^n$ & usual $T^n \curvearrowright (\mathbb{C}^*)^n$ by $(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$ extends to all of V }
so orbits provide Lagr tori

Think of toric variety as (partial) compactification of $(\mathbb{C}^*)^n$, e.g. $\mathbb{CP}^n$.

Main players & results

$$X = Bl_{H\times 0} V\times\mathbb{C}$$
$$\downarrow x \quad \downarrow y$$

$$\cup$$

$X^0 = X \setminus$ anticanon divisor } has Lagr torus fibration

(1) Y^0 = SYZ mirror to X^0 (superpotential: correction terms for discs in X^0 that we didn't have in X^0)

(2) (Y^0, W_0) = " to X

Superpot on X allows for additional noncpt Lagrus we don't see in $\mathcal{F}uk(X^0)$, those limits get $\mathcal{F}uk(X, W)$ which parametrizes SYZ mirror

(3) (Y, W_0^H) = "generalized SYZ" mirror to H
↑ add back in "central fiber" to fibration $W_0: Y^0 \to \mathbb{C}$

A little more about third statement:

we claim the following map is an equivalence

$$(\star) \quad \mathcal{F}(H) \xrightarrow{\simeq} \mathcal{F}(X, W^\vee)$$

$W^\vee$ = the coordinate on $\mathbb{C}$

$$\begin{array}{c} X \\ \downarrow W^\vee \\ \mathbb{C} \end{array}$$

- has Morse-Bott singularities (as opposed to Morse in the case of Lefschetz fibrations)

- $\mathrm{crit}(W^\vee) = \tilde{V} \cap E \simeq H$

  - $\tilde{V}$: pts in H have $y=0$ & $v=0$
  - E: $\mathbb{P}^1$-bundle over $H \times 0$
  - (see next page for $\tilde{V}$, E notation)

Then $(\star)$ takes a Lagr in $\mathrm{crit}(W^\vee) \simeq H$ & parallel transports it to get a thimble, ie element of $\mathcal{F}(X, W^\vee)$, w/ correct behavior @ ∞ ie under $W^\vee$ can only $\to \infty$ along positive real axis.

Then $\mathcal{F}(X, W^\vee)$ has a limit of cpct Lagrns to noncompact one, now allowed because of the superpotential. This corresponds to getting all of Y by, in these limits, having the last coordinate $\to 0$, so we add back in the central fiber to Y.

What is X^0 & how do we get Lagr torus fibration?

① $\mathcal{L}$ = line bundle

section = f

V = toric variety

hypersurface $H = f^{-1}(0)$

② Blowup $V \times \mathbb{C}$ along submanifold $H \times 0$ means replace $H \times 0$ by the projectivization of its normal bundle (so e.g. point in n-dimensions gets replaced by a $\mathbb{CP}^{n-1}$)

- normal bundle to V ("vanishing of" section of $\mathcal{L}$) is $\mathcal{L}|_H$
- " " to 0 in $\mathbb{C}$ is trivial line θ

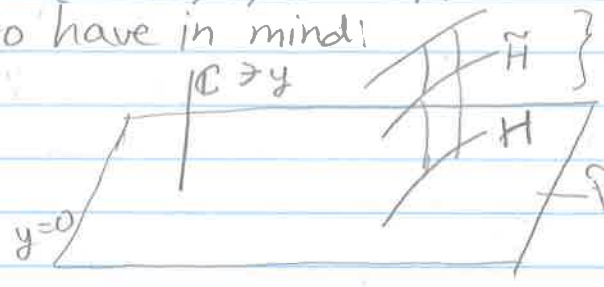
$$\Rightarrow X = \{ (x, y, (u:v)) \in \mathbb{P}(\mathcal{L} \oplus \theta) \mid f(x)v = yu \}$$

x in V, y in $\mathbb{C}$, $(u:v) \in \mathbb{P}((\mathcal{L} \oplus \theta)_{|x,y})$

$X \xrightarrow{p} V \times \mathbb{C}$ by proj^n onto 1st two coords

- $E = p^{-1}(H \times 0)$ exceptional divisor ($\mathbb{P}^1$'s worth when $x \in H$, $y = 0$)
- $\tilde{H} = \{$ pts in image of section of $E \to H \times 0$ assigning $\mathbb{P}^1$ point $(0:1)$ in each fiber $\}$
- $\tilde{V} = \{ (x, 0, (1:0)) \mid x \in V \} = \overline{p^{-1}(V \times 0 \setminus H \times 0)}$

Picture to have in mind:



V has torus action, what about $\mathbb{C}$-factor? Have S^1-action:

$$\theta \cdot (x, y, (u:v)) = (x, e^{i\theta} y, (u : e^{i\theta} v))$$

$\tilde{V} \sqcup \tilde{H}$ = fixed point strata

skip

Recall that we would like an $(n+1)$-form globally holomorphic to do SYZ. So remove anticanonical divisor:

$$(\mathbb{C}^*)^n \simeq V^\circ \subset V \qquad (\text{def}^n \text{ of toric var})$$

$$V \setminus V^\circ = D_V \quad \leftarrow \text{toric divisors compactifying } V^\circ \text{ to } V$$

$$\Omega_{V \times \mathbb{C}} = i^{n+1} \prod_j d\log x_j \wedge d\log y$$

($d\log x_j$: poles on D_V; $d\log y$: pole on $V \times 0$)

So $p^* \Omega_{V\times\mathbb{C}}$ has poles on $\tilde{V} \cup p^{-1}(D_V \times \mathbb{C})$.

Let $D = \tilde{V} \cup p^{-1}(V \setminus V^\circ \times \mathbb{C})$. Then define

1/3 third

$$X^\circ = X \setminus D = \{(x,y,z) \in \mathcal{O} \oplus \mathcal{L} \mid f(x) = yz\}.$$

(x in V°)

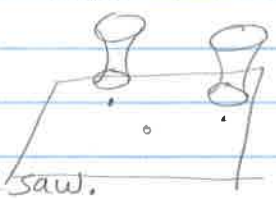
conic bundle

e.g. $H = pt \subseteq V = \mathbb{C}$ then we can take section of trivial bundle on $\mathbb{C}$ to be $x-1 : \mathbb{C} \to \mathbb{C}$ which vanishes at 1.

$X = \{(x,y,z) \in \mathbb{C}^* \times \mathbb{C}^2 \mid x - 1 = yz\}$

S^1-action here $z = u/v$ so $\theta \cdot z = u/e^{i\theta} v = e^{-i\theta} z$

and S^1-action is $(x, e^{i\theta} y, e^{-i\theta} z)$ as we saw.



Constructing Lagr torus fibration

$$\pi : X^\circ \longrightarrow \mathbb{R}^n \times \mathbb{R}_+$$

The map to final coordinate is

$$\mu_X : X \longrightarrow \mathbb{R}$$

moment map for Hamilt S^1-action on X.

Symplectic form on X

Toric variety comes w/ data of a sympl form so $(V \times \mathbb{C}, \omega_{V\times\mathbb{C}})$. We know

$$X \subseteq \underbrace{\mathbb{P}(\mathcal{L} \oplus \mathcal{O})}$$

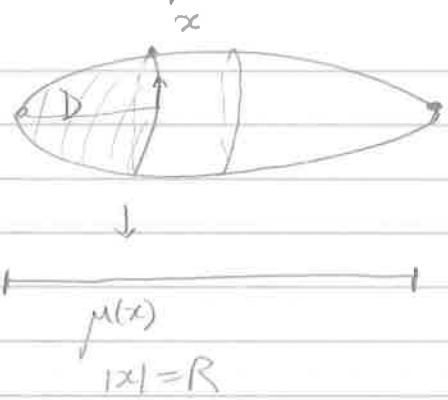
comes w/ Kähler form $\frac{i}{2\pi}\partial\bar{\partial}\log(\text{fiberwise metric on } \mathcal{L}\oplus\mathcal{O}|^2)$

Away from exceptional divisor $E = p^{-1}(H\times 0)$, X can be identified w/ $V\times\mathbb{C}$. So equip X w/ ω_ε which interpolates b/w

$p^*\omega_{V\times\mathbb{C}}$ away from E

$p^*\omega_{V\times\mathbb{C}}$ + form on $\mathbb{P}(\mathcal{L}\oplus\mathcal{O})$ near E
& so fibers $E \to H\times 0$ have suff small area

Recall μ_x is area, e.g.



$d\mu$ (radial)
ω (radial)
$\omega(\partial_\theta$, radial)

$\mathbb{CP}^1$

$$\int_D \omega_{\mathbb{CP}^1} = \int_0^R d\mu = \mu(R)$$

Here $\mu_x(x,y) = \int_{D(x,y)} \omega_\varepsilon$. ← bdd by orbit $\{(x, e^{i\theta}y)\}$ in X°

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Back to constructing $X^\circ \to \mathbb{R}^n \times \mathbb{R}_+$.

Key point $\mu_x^{-1}(\lambda)/S^1$ can be identified with V°, which then has a natural map to $\mathbb{R}^n$:

$$\text{Log}: (\mathbb{C}^*)^n \to \mathbb{R}^n$$
$$(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$$

$\mu_x^{-1}(\lambda)/S^1 \simeq V^\circ$: Recall ω_ε interpolates b/w 2 forms near & away from E.
Similarly for μ_x: $\pi|y|^2$ away from E & $\pi|y|^2 + \varepsilon \frac{|y|^2}{|f(x)|^2+|y|^2}$ near E.

- we removed case where ~~$f(x)=0$~~ $\lambda = 0$
- near E ($y=0$, $f(x)=0$) we have $|y|$ & $|f(x)|$ small so μ_x close to E.

$\widetilde{H} \subset \mu_X^{-1}(\varepsilon)$

- μ_X strictly increasing in $|y|$ for fixed x so $p^{-1}(\{x\} \times \mathbb{C}) \cap \mu_X^{-1}(\lambda)$ contains unique S^1-orbit with $|y| =$ constant.

$$p^{-1}(\{x\} \times \mathbb{C}) = \{(x, y, (u,v))\} \subset X$$
$$\uparrow$$
$$\mathbb{C}$$

Get S^1-orbit at each $|\lambda| > 0$ value.

- Get bijection b/w pts $x \in V$ and $X_{red,\lambda}$ by

$$\begin{array}{ccc} x & \longrightarrow & \left(\begin{array}{c}\text{unique } S^1\text{-orbit} \\ p^{-1}(\{x\} \times \mathbb{C}) \text{ at} \\ \text{"height" } \lambda\end{array}\right) \Big/ S^1 \\ \uparrow & & \uparrow \\ V & \cong & X_{red,\lambda} = \mu_X^{-1}(\lambda)/S^1 \end{array}$$

=

Goal V° has Lagr tori by $S^1(r_1) \times \cdots \times S^1(r_n)$ because ω_V is defined to be invariant under torus action & $S^1(r_1) \times \cdots \times S^1(r_n)$ are orbits of this action, hence ω_V vanishes on them.

How to translate this to $X_{red,\lambda}$? Kähler form induced on $X_{red,\lambda}$ doesn't have properties we want. Answer: family of maps

$$\phi_\lambda : (X_{red,\lambda}, \omega_{red,\lambda}) \longrightarrow (V, (\text{toric kähler})_\lambda)$$
$$\text{cohomologous}$$

& then get Lagr torus fibration.

$$\pi_\lambda = \text{Log} \circ \phi_\lambda : X^\circ_{red,\lambda} \longrightarrow \mathbb{R}^n$$
$$\uparrow (\mu_X^{-1}(\lambda) \cap X^\circ)/S^1$$

Putting all the $\lambda > 0$ levels together, we get

$$\pi : X^\circ \longrightarrow \mathbb{R}^n \times \mathbb{R}_+$$

$$p \longmapsto \left(\pi_{\mu_X(p)}(\mathrm{orb}_{g^t}(p)), \mu_X(p) \right)$$

tropical Background

The mirror Suppose $f = \sum_{\alpha \in A \subseteq \mathbb{Z}^n} c_\alpha x^\alpha$ (OVER for example)

Get family of hypersurfaces

2/3 thru

$$f_\tau = \sum_{\alpha \in A} c_\alpha \tau^{p(\alpha)} x^\alpha = 0. \quad \leftarrow \text{means } x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

Rewrite $|x|$ as $\tau^{-\xi}$, where τ is a small real number and ξ is a vector so $(|x_1|, \ldots, |x_n|) = (\tau^{-\xi_1}, \ldots, \tau^{-\xi_n})$. Then $|x^\alpha| = \tau^{-\langle \alpha, \xi \rangle}$ and

$$f_\tau = \sum_{\alpha \in A} c_\alpha \tau^{p(\alpha) - \langle \alpha, \xi \rangle}$$

As $\tau \to 0$, the dominating term has exponent (neg of)

$$\varphi(\xi) = \max \{ \langle \alpha, \xi \rangle - p(\alpha) \mid \alpha \in A \}$$

The "vanishing" of $\varphi(\xi)$ is where there is a non-unique max, because this is the limit of 2 terms in f_τ which can cancel to give zero.

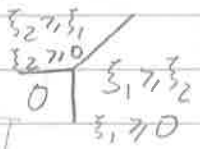
* ① First do $V(f = 1 + x_1 + x_2) \subset (\mathbb{C}^*)^2$, $A = \{(0,0), (1,0), (0,1)\}$, $p = 0$, $\varphi(\xi) = \max\{0, \xi_1, \xi_2\}$

e.g. ~~$f(x_1, x_2) = 1 + x_1 + x_2 + x_1 x_2$~~

② $A = \{(0,0), (1,0), (0,1), (1,1)\}$.

$p(0,0) = p(1,0) = p(0,1) = 0$, $p(1,1) = 1$

$\Rightarrow f_\tau = 1 + x_1 + x_2 + \tau x_1 x_2 : (\mathbb{C}^*)^2 \to \mathbb{C}$



$$\varphi(\xi_1, \xi_2) = \max\{0, \xi_1, \xi_2, \xi_1 + \xi_2 - 1\}$$

$\xi_1, \xi_2 \leq 0 \Rightarrow 0$ is max

$\xi_1 \geq \xi_2$ & $1 \geq \xi_2 \Rightarrow \xi_1$ max similar with ξ_2

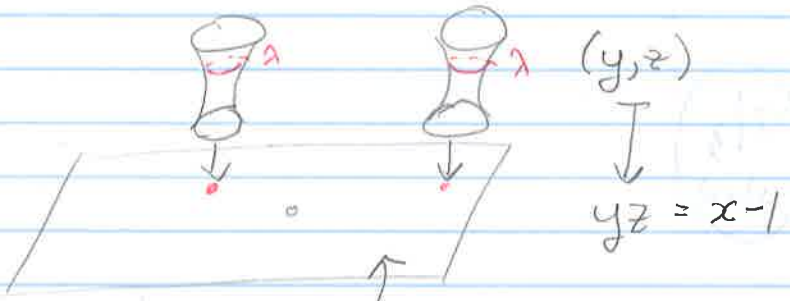
$\xi_1 \geq 0$

example $Bl_{pt}\,\mathbb{C}^2 \supset X^\circ = \{(x,y,z) \in \mathbb{C}^* \times \mathbb{C}^2 \mid yz = x-1\}$

$$\downarrow p$$

$$V^\circ \times \mathbb{C}$$

$p_V = pr_V \circ p : (x,y,z) \mapsto x$

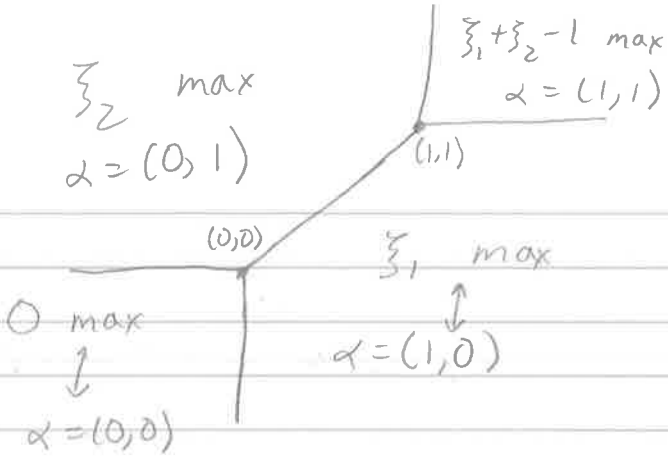


To get Lagr torus fibration: $V^\circ = \mathbb{C}^*$ has Lagr tori given by $S^1(r)$. We know the moment map is $|x|^2 - |y|^2$ from last time (up to scalar).

So $p^{-1}(\{x\} \times \mathbb{C}) \cap \mu_x^{-1}(\lambda)$ is the unique S^1-orbit over x at height λ.

So $X_{red,\lambda} = \mu_x^{-1}(\lambda)/S^1$ can be identified with V° by each orbit at height λ corresponding to point it's in fiber over.

The upshot is: Lagrangian tori are obtained by lifting $S^1(r) \subset V^\circ$ to the reduced spaces $X_{red,\lambda}$ and then spinning by the S^1 action to get tori Denis mentioned last time. Ie this will be a fiber of $\pi : X^\circ \to \mathbb{R} \times \mathbb{R}_+$.



(ξ_1, ξ_2)-plane lines are "vanishing" of $\varphi(\xi)$ and regions labelled by α that achieves maximum

$$(x_1, \dots, x_n) \xrightarrow{\text{Log}} \frac{1}{|\log \tau|}(\log|x_1|, \dots, \log|x_n|)$$

$\cap$

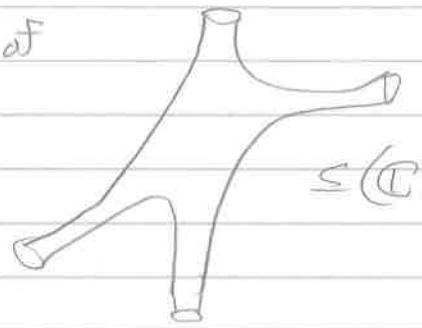
H_τ

$$= \pm(\xi_1, \dots, \xi_n)$$

As $\tau \to 0$, $\text{Log}(H_\tau) \to$ tropical picture above.

In this example get family of

(and mirror is $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{CP}^1 = Y$)



Back to description of the mirror: there are 2 ways.

① Y described by polytope $\Delta_Y \subseteq \mathbb{R}^{n+1}$

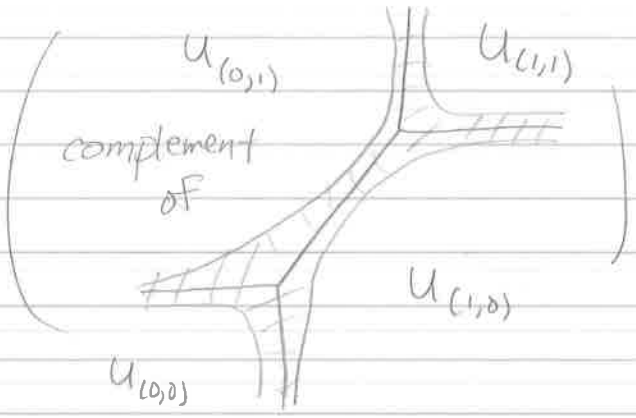
$$\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta \geq \varphi(\xi)\}.$$

e.g. Δ_Y in above case ① would be in $\mathbb{R}^3$ where
$\eta \geq 0$ everything above $\eta = 0$ plane when $\xi_1, \xi_2 \leq 0$
$\eta \geq \xi_1$ in right region, $\eta \geq \xi_2$ in left, &
$\eta \geq \xi_1 + \xi_2 - 1$ in upper corner.

From polytope we can recover toric variety Y.

② Glue via charts ← easier to see as SYZ mirror to X this way.

Skip ② if out of time



$\times \mathbb{R}_+ = B^{reg}$

$= \bigsqcup U_\alpha$

↑ actually = region in picture $\times \mathbb{R}_+$

Dualize U_α to $U_\alpha^\vee$ as Denis explained:

$b^\circ \in U_\alpha$ reference point, L° = fiber with homology basis $(\gamma_1, \dots, \gamma_n, \gamma_\delta)$ in H_1, then nearby fibers have coordinates via ① holonomy and ② area (measured by cylinders from $\|$ transporting γ_i) from base Lagrangian.

Over $(S_1, \dots, S_n, \lambda) \in U_\alpha$, coordinates on $U_\alpha^\vee$ are

$$(L, \nabla) \mapsto (v_{\alpha,1}, \dots, v_{\alpha,n}, w_{\alpha,0}) = (T^{S_1} \underbrace{\nabla\gamma_1}_{\text{holon.}}, \dots, T^{S_n}\nabla\gamma_n, T^\lambda \mathrm{Vol}(\delta))$$

Ultimately we glue charts and get the toric variety described by ①. Note: $w_{\alpha,0} \neq 0$ here and we get Y° this way. Y is obtained by adding in $w_0^{-1}(0)$ fiber.

↳ eg • mirror for $\mathbb{C}^2 \to \mathbb{C}$ $(x,y) \mapsto xy$ is as previously described

• for p.o.p $1 + x_1 + x_2 = 0$ get $\Delta_Y = \{(\xi,\eta) \mid \eta \geq \max\{0, \xi, \xi_2\}\}$ & under $(\xi_1, \xi_2, \eta) \mapsto (\eta - \xi_1, \eta - \xi_2, \eta)$ which gives polytope $\Delta = (\mathbb{R}_{\geq 0})^3$. This gives toric variety $\boxed{\mathbb{C}^3 = Y}$

OVER

(ie one cone so one affine chart $\operatorname{Spec} \mathbb{C}[e_1, e_2, e_3] = \mathbb{C}^3$

$z \geq \max\{0, \xi_1, \xi_2\} \Rightarrow z - \xi_1, z - \xi_2, z \geq 0$)

- Y for $f = c + x_1 + x_2 + x_1 x_2$ is

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

$$\downarrow$$

$$\mathbb{CP}^1$$