Contents

1 A-side: Fukaya category **3**
1.1 Objects . . . 3
1.2 Homological algebra . . . 3

2 Computations for the genus 2 surface **5**
2.1 Finding the split-generators . . . 5
2.2 Reduce to one Lagrangian on an orbifold . . . 7
2.3 Computing structure maps μ^k on Floer complex for an orbifold . . . 7
2.3.1 Floer complex for immersed Lagrangian in the case when it comes from a group action . . . 7
2.3.2 The grading . . . 8
2.3.2.1 Dimension of moduli spaces. . . . 9
2.3.2.2 Deformations of the A_∞-structure . . . 9
2.4 The boundary of a J-holomorphic curve must be contractible in $\Gamma := H_1^{orb}$. . . . 9
2.5 Calculations . . . 10
2.5.0.3 μ^1. . . . 10
2.5.0.4 μ^2. . . . 10
2.5.0.4.1 What "curve through $e, q \in CM$" means. . . . 11
2.5.0.4.2 Constant triangles. . . . 12
2.5.0.5 μ^3. . . . 12
2.5.0.6 μ^4. . . . 13
2.5.0.7 μ^5. . . . 13
2.6 First glance: finite determinacy . . . 13

3 B-side: Category of singularities **14**
3.1 Overview of the mirror and category on B-side . . . 14
3.2 Construction of the crepant resolution . . . 14
3.2.1 General definition . . . 14
3.2.2 Toric description in this case . . . 15
3.3 D^b_{sing} and its relation to matrix factorizations . . . 15
3.3.1 Definitions . . . 15
3.3.2 Relation to matrix factorizations . . . 16
3.4 The A_∞-str on $\wedge(V)$ in §11: Koszul duality . . . 17
3.5 The A_∞-str on $\wedge(V)$ in §3,4,5: leads to a classification thm . . . 18
3.5.1 Review from Jeff's talk: $HH(A, A)$ and deformation theory . . . 19
3.5.2 Finding correct deformation in $HH(A, A)$ w/ help of Kontsevich formality 19
3.5.2.1 What deformations look like in $\mathfrak{g}$. . . . 20
3.5.2.1.1 Maurer-Cartan equation for $\mathfrak{g}$ . . . 21

3.6 Conclusion 21
3.6.1 Finite determinacy 21
3.6.2 Equivalence of derived categories 22

4 References **22**

Talk on Seidel's HMS for genus 2 curve paper, given by Catherine Cannizzo.

1 A-side: Fukaya category

1.1 Objects

The Fukaya category is a $\mathbb{Z}/2$-graded A_∞-category over $\mathbb{C}$. It consists of balanced curves, i.e. simple closed curves L where

$$\int_L \sigma^* \theta = 0$$

where $\sigma : L \to S(TM)|_L$ sends a point in L to the oriented unit tangent vector of L at that point, and θ satisfies $d\theta = \pi^* \omega$ where $\pi : S(TM) \to M$. (This exists as follows: the tangent bundle is a complex line bundle which has nonzero degree $2 - 2g = -2$ because K_M has degree $2g - 2$. Then show a closed 2-form should be exact.) We can equivalently characterize nullhomologous balanced curves as follows:

Claim 1.1.1. A nullhomologous curve is balanced iff it divides M into $M_\pm$ such that

$$\chi(M_+)/area(M_+) = \chi(M_-)/area(M_-)$$

Objects of the Fukaya category are balanced curves L equipped with 1) an orientation and 2) a Spin structure. Recall: a Spin structure is an equivariant lift of the SO_n-frame bundle to $Spin(n)$ where $Spin(n) \to SO_n$ is a double cover. (Equivariant means: the projection on bundles induced from $Spin(n) \to SO(n)$ respects the actions of each group on the corresponding bundle.) The obstruction to such a lift is the second Stiefel-Whitney class $w_2(L)$. Pin structures are classified by $H^1(L; \mathbb{Z}/2)$.

The balanced condition is an analogue of monotonicity (Maslov index proportional to symplectic form on discs bounded by a single Lagrangian) but when the boundary is in several Lagrangians.

1.2 Homological algebra

Ultimately we are going to state a derived equivalence of split-closed derived categories, showing that the symplectic side (A-model) and complex side (B-model) are both equivalent to the same thing. Let's recall what the split-closed derived category is.

- Start with a $\mathbb{Z}/2$-graded A_∞-category $\mathcal{B}$ (e.g. $\mathcal{F}(M)$).

- Recall the Yoneda embedding $\mathcal{B} \to mod(\mathcal{B})$ sending an object Y to the A_∞-module $\mathcal{Y}$ (which, by definition, is a functor from $\mathcal{B}^{opp}$ to chain complexes, and here for object X we get the chain complex $\hom_{\mathcal{B}}(X, Y)$). This functor is full and faithful on the cohomological level (i.e. replacing chain complexes with their cohomology).

  Recall faithful/full means injective/surjective on hom sets. Note that full does not mean surjective on objects.

- Taking cohomology on the target $mod(\mathcal{B})$ gives the triangulated category $Mod(\mathcal{B})$. It is split-closed.

  Definition 1.2.1. A category is *split-closed* if any endomorphism of an object that squares to itself (i.e. idempotent, think of like a projection) allows us to write that object as a direct sum of two other objects in the category.

- Take the smallest full subcategory of $Mod(\mathcal{B})$ containing the image of the Yoneda embedding, is triangulated, and split-closed.

- This is the split-closed derived category of $\mathcal{B}$ and denoted $D^\pi(\mathcal{B})$.

- We say that the objects of a full A_∞-subcategory $\mathcal{A}$ *split-generate* $\mathcal{B}$ if every object of $\mathcal{B}$ is isomorphic to one in $D^\pi(\mathcal{A})$.

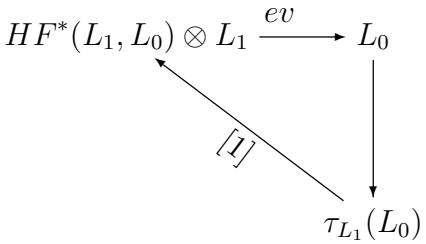
So how can we determine when objects split-generate in the Fukaya category?

Lemma 1.2.2 (Lemma 6.4)**.** *$\{L_1, \ldots, L_r\}$ objects of the Fukaya category, equipped with non-trivial Spin structures. Suppose another object L_0 is isotopic (with reversed orientation) to:*

$$\tau_{L_r} \ldots \tau_{L_1}(L_0)$$

Then L_0 is split-generated by $\{L_1, \ldots, L_r\}$.

In particular, recall the triangle



where the top map evaluates a morphism on L_1 (direct sum of copies of L_1, one for each generator of $HF^*(L_1, L_0)$ and ev is the morphism given by whichever generator we pick that

defines the corresponding summand – recall a morphism is an intersection point). In particular, we can compose the vertical arrows to obtain a morphism

$$L_0 \to \tau_{L_r} \dots \tau_{L_1}(L_0) \cong L_0[1]$$

so we get an element of $H^1(L_0; \mathbb{C})$.

If the map itself is isotopic to the identity, then the L_i split-generate.

Lemma 1.2.3 (Lemma 6.5)**.** *Suppose $\tau_{L_r} \dots \tau_{L_1}$ is isotopic to the identity. Then $\{L_1, \dots, L_r\}$ split-generate $\mathcal{F}(M)$.*

2 Computations for the genus 2 surface

Denote the genus 2 surface by M. To do computations, we will envision it in three different ways: one to get the split-generators, one to reduce computations to one Lagrangian (this one is the primary way to think about it), and one to do those computations equivariantly on the Lagrangian.

2.1 Finding the split-generators

To determine the split-generators, we consider the genus 2 surface M as a double-branched cover of the sphere, as follows. Consider the 6th order polynomial

$$P(z) = z(z-1)(z-\zeta)(z-\zeta^2)(z-\zeta^3)(z-\zeta^4)$$

where ζ is a fifth root of unity. To define the square-root of this, we need two domains, one on which we define the function to be $\sqrt{P(z)}$ and one $-\sqrt{P(z)}$. In particular, if we go around an even number of branch points, the function $\sqrt{P}$ remains unchanged, but if we go around an odd number, we pass through a branch cut and must pass to the other surface.

[draw picture of

1) branch points and branch cuts on the two spheres
2) opening up the branch cuts and gluing to the genus 2 surface
3) the projection to S^2 by identifying the corresponding two points in each sphere, so we can see the branching happening over 6 points where the two identified points are the same because they get glued.

See end of document for pictures.]

Plan of action:

1. Draw five curves on S^2.

2. Explain why they lift to five Lagrangians in the Fukaya category on M.

3. Explain why these five split-generate.

Here are the five curves. [draw picture of star] Note that to project down to S^2, we want to consider Lagrangians on the genus 2 surface that are invariant under the "hyperelliptic involution" i.e. $z \in \mathbb{CP}^1$ on the domain for $\sqrt{P}$ maps to $z \in \mathbb{CP}^1$ on the domain for $-\sqrt{P}$. In particular, consider the following [show curve that projects to L_1]. Note that a Dehn twist along the lifted L_1 corresponds to only a half twist on the base S^2.

Lemma 2.1.1 (Lemma 10.1)**.** *The curves $\{L_1, \ldots, L_5\}$, lifted to M and equipped with non-trivial Spin structures, split-generate $D^\pi \mathcal{F}(M)$.*

Sketch. We want to relate this to a group that we know something about and can do calculations with, the braid group. But Dehn twists only satisfy the relations of a braid group when they intersect in only one place. There are too many intersections among the L_i so instead we look at the following four K_i [show picture]. The claim is that these split-generate using the second lemma on split-generation above, and then the K_i are split-generated by the L_i using the first lemma above.

The Dehn twists τ_{K_i} now satisfy the relations of the braid group Br_5 with 5 strands. [show picture of braid generators].

$$Br_5 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2 \rangle$$

So setting $\sigma_i = \tau_{K_i}$, it can be show that the relations are satisfied. This is because Dehn twists commute when there is no intersection point, otherwise with one intersection point they satisfy the first relation. The K_i's have these required intersections.

The upshot is that we get a homomorphism to the mapping class group:

$$\phi : Br_5 \to MCG(M)$$

and the kernel is isomorphic to $\mathbb{Z}$, generated by the central element Δ^4 where $\Delta^2 = (\tau_{K_4} \ldots \tau_{K_1})^5$. In particular we get that $(\tau_{K_4} \ldots \tau_{K_1})^{10}$ is isotopic to the identity, so by the second lemma the K_i split-generate. A calculation shows that $\tau_{L_5} \ldots \tau_{L_1}(K_2)$ is isotopic to $K_2[1]$ and similarly for the other K_i's, so at last we get that the L_i split-generate.

□

2.2 Reduce to one Lagrangian on an orbifold

Note that the other L_i are obtained from L_1 by rotation by $e^{2\pi i/5}$. So we can quotient by this action on M (by quotienting by the action on each sphere and gluing together), and then just work with the one Lagrangian that is the image of each L_i. There is a technicality in that we get three "orbifold points" where this $\mathbb{Z}/5$ action is not free. Zero is a fixed point on each sphere, and zero is also a branch point so only gives rise to one fixed point on M. However ∞ in each sphere is also fixed, and they do not get identified, so they give rise to two more orbifold points, for a total of three. Here is how to see what the image of any of the Lagrangians L_i looks like [show picture].

Denote the orbifold sphere by $\bar{M}$ and the single Lagrangian on it by $\bar{L}$.

2.3 Computing structure maps μ^k on Floer complex for an orbifold

2.3.1 Floer complex for immersed Lagrangian in the case when it comes from a group action

When computing the μ^k, we will ultimately be counting polygons in a given moduli space. But we will be able to determine that some of these moduli spaces are empty by an orbifold analogue of the fact that a pseudo-holomorphic map u must have boundary mapping to a curve that is null-homologous, i.e. we can't have a map u that goes through a hole.

The orbifold homology of this orbifold sphere $\bar{M}$ is the group $\Gamma = H_1^{orb}(\bar{M}) \cong (\mathbb{Z}/5)^3/\langle\Delta\rangle$ where Δ is the diagonal element $(1,1,1)$. The generators correspond with winding around an orbifold point. So for example, $(1,1,0)$ is the same thing as $(0,0,-1)$ because they sum to zero in Γ.

A usual sphere has trivial H^1. Here, because the orbifold points encode a $\mathbb{Z}/5$ singularity, we need to go five times around any given point to get zero, hence the factors of $\mathbb{Z}/5$. Going around all three also gives a zero (think of removing the three orbifold points, then we can contract such a curve to a point). In particular, recall in the construction above that we go around all three orbifold points. Thus the given Lagrangian is trivial in orbifold homology. This means (because the Lagrangian is fixed under rotations about the orbifold points), we have an action on the generators of the Floer complex by this orbifold homology group Γ. In fact we can think of Γ as acting on the universal abelian cover X by rotations about each orbifold point down below so deck transformations up top, subject to the relation that rotation around all three is trivial, which gives us an action on X. This realizes the third way to visualize what's going on: $\bar{M} = X/\Gamma$. Note that in Seidel's Remark 9.2, the M there is X here. I.e. the M there is not the genus 2 curve but the universal abelian cover.

Now we come to the definition of the Floer complex for $\bar{L}$:

$$CF^*(\bar{L},\bar{L}) = \bigoplus_{\gamma\in\Gamma} CF^*(L,L)^\gamma$$

where $CF^*(L,L)^\gamma = CF^*(L,\gamma(L))$ and $CF^*(L,L) = CM^*(f) = \mathbb{C}e \oplus \mathbb{C}q$ is the Morse complex for a Morse function on L with one maximum and minimum. Note that the minimum is suggestively called e because it will act like the identity when we compute μ^2. This description allows us to explicitly see the *weights* of a generator, namely the γ such that an intersection lies in $CF^*(L,L)^\gamma$.

For an immersed Lagrangian, the definition of the Floer complex is that we get the two Morse generators plus two points at each self-intersection point (since perturbing a self-intersection point a little gives two intersection points. Think of an x and y axes and then another axes pushed off a little – the pushed off picture intersects the original picture in two points).

2.3.2 The grading

Recall that to get a $\mathbb{Z}$-grading we need twice the first Chern class to vanish. However, here the genus is $2-2g$ so the Euler characteristic is -2, not zero. What we do instead is equip the pair of pants (i.e. orbifold sphere $\bar{M}$ minus the three orbifold points) with fractional 1/3-gradings.

Consider the tensor η on (S^2, z) given by $z^{-2}dz^3$, or on the orbifold sphere $(\bar{M}, w)$ where $z = w^5$ this becomes $5^3 w^2 dw^3$. This is a section of $(T^*\bar{M})^{\otimes 3}$ which has a double zero at each orbifold point. If we remove these orbifold points, we get a map $L \to S^1$ defined by evaluating η on a basis and normalizing it to be unit.

Since Seidel later defines D to be the set of points where Γ does not act freely and Γ is something which acts on the universal cover X, I think what he calls M in the gradings section §8 is actually the abelian universal cover X, but I'm not certain about this. I will use M to be consistent with Seidel, but it may actually be what I previously called X. Then we can define $\mathcal{F}(M,D)$ to consist of objects that are balanced Lagrangians which avoid D (D is the lift of the three orbifold points to M, i.e. where the action of Γ is not free) and that come equipped with a lift a of the above map to $\mathbb{R}$, which is by definition a 1/3-grading on the Lagrangian. It is a fact that $\mathcal{F}(M,D) \to \mathcal{F}(M)$ is full and faithful.

If two Lagrangians intersect at an angle α (which in our case $\alpha = 2\pi/5$), then define the index on an intersection point x to be

$$i^3(x) = \frac{3\alpha + a_1(x) - a_0(0)}{\pi}$$

where a_i are lifts $L_i \to \mathbb{R}$.

2.3.2.1 Dimension of moduli spaces. We are only interested in moduli spaces of virtual dimension zero (i.e. Fredholm index of a given curve u plus dimensions coming from the marked points quotiented by automorphism group, is 0) because these are the moduli spaces we are counting in the structure maps. We have the following analogue of dimension in this case: the dimension of $\mathcal{M}(x_0, \ldots, x_d)$ being zero means

$$i^3(x_0) - \sum_{j=1}^{d} i^3(x_j) = 3(2-d) + 2\sum_{z \in D} 2\deg(u, z)$$

where i is the fractional grading. The factor of 3 comes from the fractional grading, the 2 inside the right-hand summand comes from being $\mathrm{ord}(\eta, z)$, i.e. we have a double zero at each point in D. In the usual case, we replace the i^3 with i, get rid of the factor of 3 multiplying $2-d$ and get rid of the sum on the right (so that Fredholm index, i.e. the left hand side, plus $d-2 = (d+1)-3$ the dimension from having $d+1$ marked points once we've fixed 3 of them, is zero).

In the orbifold case described above when we to find $\mathcal{M}(\bar{x}_0, \ldots, \bar{x}_d)$ for elements $\bar{x}_i$ with weights γ_i, this is only nonempty when $\gamma_0 = \gamma_1 \ldots \gamma_d$ (so in our case the sum of the weights of the input points must equal the weight of the output point, i.e. their difference is trivial in orbifold homology) and in this case it is

$$\mathcal{M}(x_0, x_1, \gamma_1(x_2), \ldots, \gamma_1 \ldots \gamma_{d-1}(x_d))$$

2.3.2.2 Deformations of the A_∞-structure When we remove the problems points, $\bar{D}$ on the orbifold sphere, we have a pair of pants. This has the usual structure maps encoded by μ_0 (here the subscript denotes the order of the deformation). But what about counting polygons that go through the orbifold points? The deformed structure maps on $\bar{M}$ become as follows:

$$\mu^i = \mu_0^i + \mu_1^i + \ldots$$

where μ_j^i inputs i elements, and then counts the number of polygons *that go through j orbifold points, with ramification 5 each.* The reason it has to be ramification 5 is so that we can lift it to the genus 2 curve. In particular, in this case

$$\deg(\mu_k^i) = 6 - 3i + 4k$$

We will use this to eliminate possibilities in the coming calculations.

2.4 The boundary of a J-holomorphic curve must be contractible in $\Gamma := H_1^{orb}$.

We will go into more detail of why $\gamma_0 = \gamma_1 \ldots \gamma_d$, which will allow us to conclude that certain moduli spaces are empty. Suppose we want to consider the moduli space $\mathcal{M}(\bar{x}_0, \ldots, \bar{x}_d)$ in

some A_∞-structure map, where $\bar{x}_i$ has weight γ, i.e. it is an element of $CF(L, \gamma(L))$ where L is the lift to the abelian universal cover and $\gamma \in \Gamma = (\mathbb{Z}/5)^3/\langle\Delta\rangle$. That is, the moduli space of (virtual dimension zero) pseudo-holomorphic maps that go through $d+1$ points. Here "virtual dimension 0" means Fredholm index plus dimension from the marked points minus the dimensions from stabilizers, all suitably adjusted in this setting from the $\mathbb{Z}/5$ action and fact that $2c_1$ is not zero. And we want virtual dimension zero for the reasons mentioned above – they are the only ones that we count in the sums in the structure maps.

Working on the orbifold X/Γ should be the same as working with the lifted curves on X. In particular, the lifted u should have contractible boundary. Let's unravel what this means. A curve u that goes through the $d+1$ points down below should lift to a curve that goes through $d+1$ points on a lift to the universal cover. In particular, since we should get that the boundary of the lifted u on X is a closed loop (which will be contractible on the abelian universal cover) we require that the final Lagrangian is the first one again i.e. $\gamma_0 = \gamma_1 \dots \gamma_n$. Reasoning: we start with an intersection point of L and $\gamma_0(L)$, then we get one of $L, \gamma_1(L)$ then one of $\gamma_1(L)$ and $\gamma_1(\gamma_2(L))$ and so on.

In our situation, we have an abelian group so $\gamma_0 = \gamma_1 \dots \gamma_n$ says that the "output weight" in the structure map should be the sum of the "input weights".

2.5 Calculations

2.5.0.3 μ^1**.** First consider μ^1. This has one input and one output, thus we require $\gamma_0 = \gamma_1$ in the notation above. No two distinct generators have the same weight, so the only way to get a nonempty moduli space is if we have the same generator. Consider $\mathcal{M}(x, x)$ where x is the generator. Recall that the degree of μ^1_k is $6 - 3 \times 1 + 4k = 3 + 4k$. The most that two indices can differ is 3 so we must have $k = 0$. But then μ^1 should change the degree by 3, which cannot happen when the input and output are the same. So μ^1 is zero.

2.5.0.4 μ^2**.** Now we require $\gamma_0 = \gamma_1 + \gamma_2$ and μ^2_k has degree $6 - 3 \cdot 2 + 4k = 4k = ind(p_0) - ind(p_1) - ind(p_2)$. Note that because indices are between 0 and 3:

$$3 \geq ind(p_0) - \sum_{i=1}^{r} ind(p_i) = 6 - 4r + 4k \geq -3r$$

So here $r = 2$ and we must have $k = 0$. Thus we only need to count pseudo-holomorphic triangles in the pair of pants, i.e. there are none passing through orbifold points.

There are constant and non-constant triangles. The constant triangles must satisfy $\gamma_0 = \gamma_1 + \gamma_2$ and similarly for the indices. At first glance, one would think that a constant triangle should

have the two inputs and output point all be the same generator. But recall that $x_i, \bar{x}_i$ are the same underlying point, and we also need to say what it means to go through e, q. In particular, given the requirement on the weights, it would not work to have moduli spaces with all x_i or all $\bar{x}_i$.

2.5.0.4.1 What "curve through $e, q \in CM$" means. We consider curves u that are pseudo-holomorphic triangles, so we have three marked points ζ_i mapping to two input points p_1, p_2 and one output point p_0. For example $\mu^2(e, q)$ is a map on $CM^*(f) \otimes CM^*(f) \to CM^*(f)$. The following are definitions of moduli spaces needed for μ^2 when at least one point is a Morse critical point. Note that in a moduli space $\mathcal{M}(p_0, p_1, p_2)$ we denote the output point by p_0 and the two input points by p_1, p_2. So we look at the coefficients on p_0 that appear in $\mu^2(p_1, p_2)$ by defining the moduli spaces whose counts are the coefficients appearing in the sum in $\mu^2(p_1, p_2)$. Note that the three items below can be defined in general for any μ^k.

1. It is possible to have μ^2 where the inputs are intersection points and an output is a Morse critical point. This happens when the first and third Lagrangian are the same. Then the requirement on u in such moduli spaces is that $u(\zeta_0) \in W^s(p_0)$. If $p_0 = e$ then this says the triangle must go through e (because any other point flows away from the minimum so is not in the stable manifold). If $p_0 = q$ then the triangle can go to any point not e. For dimensional reasons we only get constant maps in the latter case.

2. A μ^2 with one input a Morse critical point p and other input not, means that we count moduli spaces of u that send the input point corresponding to the Morse critical point to the unstable manifold of p, $W^u(p)$. If $p = e$ is the minimum, then everything except q flows back to e and the given point can map anywhere that is not q. In this case the only relevant moduli spaces are for constant triangles. If $p = q$ is the maximum, then the point is required to map to q since nothing else flows backwards to q.

3. The definition of μ^2 when the two inputs are Morse critical points is more complicated. In this case we can think of this as a composition of two morphisms from a Lagrangian to itself. We need to know how to talk about moduli spaces where two or three of the points are in CM. They are described by patching together pseudo-holomorphic maps and Morse flow lines. This is encoded in the data of a tree as follows.

   Draw a tree with 3 semi-infinite edges. It has vertices, finite edges, and semi-infinite edges. The vertices v correspond with holomorphic maps u_v. We require any vertex to have at least 3 edges (i.e. valency $|v| \geq 3$). We want there to be three components of $\mathbb{R}^2 \backslash T$ labelled L_0, L_1, L_2 (where these are all the same Lagrangian in our case). The rule is that finite edges should be sandwiched between L_i with the same underlying curve, which is always the case here since all three are the same. The valency of a vertex gives

the number of marked points. The pseudo-holo curve u_v maps the marked points so that in between they lie on the Lagrangians which label the regions adjacent to the vertex v of the tree.

Finite edges (sandwiched between the same Lagrangian) tell us to assign a gradient flow line of f from the marked points encoded in the vertices at either end of the finite edge. At semi-infinite edges we check whether the sandwiching Lagrangians are the same or not. If they are, do the machinery of the previous item. If they are not, then we require mapping to an intersection point as usual.

For our purposes: μ^2 on CM reduces to the usual cup product on singular cohomology (under the isomorphism with Morse cohomology).

2.5.0.4.2 Constant triangles. So going back to the calculations we find the following for cases 1), 2) and 3) above. Note that we are restricted by $ind(p_0) - ind(p_1) - ind(p_2) = 0$ and similarly for the weights.

1) $\mu^2(x_i, \bar{x}_i) = q = -\mu^2(\bar{x}_i, x_i)$
2) $\mu^2(x_i, e) = x_i = -\mu^2(e, x_i)$, $\mu^2(\bar{x}_i, e) = \bar{x}_i = \mu^2(e, \bar{x}_i)$
3) $\mu^2(q, e) = q = -\mu^2(e, q), \mu^2(q, q) = 0$

Non-constant triangles. We also have a triangle at the front and at the back. For example consider the triangle with underlying points x_1, x_2, x_3. If we take generators x_1, x_2, then the output point must be $\bar{x}_3$ because looking at weights we get (weight $\bar{x}_3$) $= (1, 1, 0) =$ (weight x_1+ weight x_2) $= (1, 0, 0) + (0, 1, 0)$ and similarly on indices we get $2 = 1 + 1$. Going through the permutations we get:

$$\mu_0^2(x_1, x_2) = \bar{x}_3 = -\mu_0^2(x_2, x_1)$$
$$\mu_0^2(x_2, x_3) = \bar{x}_1 = -\mu_0^2(x_3, x_2)$$
$$\mu_0^2(x_3, x_1) = \bar{x}_2 = -\mu_0^2(x_1, x_3)$$

2.5.0.5 μ^3. The degree of μ_k^3 is $6-3\cdot 3+4k = 4k-3$, which should satisfy $3 \geq 4k-3 \geq -9$ so either $k = 0$ and it's degree -3 or $k = 1$ and the first order deformation is a degree 1 map. We also have the restriction on the weights: $\gamma_0 = \gamma_1 + \gamma_2 + \gamma_3$.

We can interpret one of the triangles above as a triangle through e, in particular an element of the moduli space $\mathcal{M}(e, x_1, x_2, x_3)$. (Recall that when the input points are intersection points we can have the output point being a Morse critical point, and in this case we require the

appropriate marked point to go to the stable manifold, so when the point is e the curve must go through e.) The only μ^3 contribution is μ_0^3 of degree -3:

$$\mu_0^3(x_3, x_2, x_1) = -e$$

2.5.0.6 μ^4**.** The degree of μ_k^4 is $6-3\cdot 4+4k = -6+4k$ where we require $3 \geq -6+4k \geq -12$, so we could have $k = 0$ with degree -6, $k = 1$ and degree -2, or $k = 2$ and degree 2. To have degree 2 we would need the output point to have index 2 more than the sum of the four inputs, and also the output weight should be the sum of the four input weights. Index of outpoint point is either 2 or 3. If it's 2 then the four input points are also zero, contradiction with the weights. If it's 3 then it's q and we must have e three times and and index 1 point: again contradiction with the weights. For degree -6 we can eliminate possibilities as well.

For μ_1^4 we count pentagons with ramification 5 at an orbifold point. There are six in the picture, each going through exactly one of x_i or $\bar{x}_i$. Their effect on μ_1^4 is that they cancel, but this is a result of the A_∞-relations, so there is no new information.

2.5.0.7 μ^5**.** Degree of μ_k^5 is $6 - 3 \cdot 5 + 4k = -9 + 4k$ and $3 \geq -9 + 4k \geq -15$. So can have degree $-9, -5, -1, 3$ corresponding to $k = 0, 1, 2, 3$. We look at μ_1^5. We take three pentagons with boundary going through e and each of which have ramification 5 at a distinct orbifold point. The map first folds a pentagon to a 1-gon by $z \mapsto z^5$. Then this is mapped to the orbifold sphere in each of the three caps. In particular, all 5 vertices get identified to one point.

$$\mu_1^5(x_1, x_1, x_1, x_1, x_1) = -\mu_1^5(x_2, x_2, x_2, x_2, x_2) = \mu_1^5(x_3, x_3, x_3, x_3, x_3) = -e$$

If the subscripts are not all the same, then those μ^5 terms vanish.

2.6 First glance: finite determinacy

To factor in all $CF(L_i, L_j)$ and not just $CF(L, \gamma_i(L))$, we multiply our above work with $CF^*(\bar{L}, \bar{L})$ by the group action. The conclusion is that:

$$D^\pi \mathcal{F}(M) \cong D^\pi(\mathcal{A} \rtimes Z)$$

where $\mathcal{A}$ is an algebra A with a deformed A_∞-structure defined later. The underlying algebra A has a usual A_∞-structure which agrees up to order 2 with the algebra given by matching the generators of the Floer complex with elements of the exterior algebra $\wedge(\mathbb{C}^3)$ as done in the table beneath Figure 7 on page 25 in Seidel's paper. (By "up to order 2" I mean that μ^1 is zero and μ^2 corresponds with wedge product. But on the usual algebra structure of $A = \wedge(V)$ there are no structure maps above 2, so we need to deform it to agree with the structure maps

in the Fukaya category.)

We claim that the relations we computed in the Fukaya catgory uniquely determine the A_∞-structure up to isomorphism. We now switch over to the complex side to see how to match up the A_∞ relations.

3 B-side: Category of singularities

3.1 Overview of the mirror and category on B-side

Remark: the mirror of the pair of pants is $(\mathbb{C}^3, -v_1v_2v_3)$. So it makes sense that it should be similar for the genus 2 curve.

First we will explain geometrically what the mirror is. Note that it will be a Landau-Ginzburg (LG) model because the genus 2 curve is not Calabi-Yau (the first Chern class is not trivial because it has nonzero degree). Then we will explain what category we take. Because the B-side is a LG model, the superpotential will need to factor in somehow into the definition of the category.

The claim is that the mirror, under homological mirror symmetry (HMS), to the genus 2 surface has the following description: it is a LG model where the manifold is the crepant resolution X of $\bar{X} := V/Z$ where $V = \mathbb{C}^3$ and $Z \cong \mathbb{Z}/5$ is generated by the diagonal matrix given by (ζ, ζ, ζ^3) where $\zeta = e^{2\pi i/5}$. The crepant resolution can be thought of as a blowup in a way that preserves the canonical class $c_1(\bar{X}) = c_1(X)$.

The category on the B-side is called the *category of singularities*. The statement of HMS in this case is that its split-closure is equivalent to that of the Fukaya category. On $W^{-1}(0)$ it's given by:

$$D^b_{sing}(W^{-1}(0)) = D^b(W^{-1}(0))/Perf(W^{-1}(0))$$

where $W : \mathbb{C}^n \to \mathbb{C}$ is a polynomial, $W^{-1}(0)$ has one singular point at zero, $D^b(W^{-1}(0))$ means bounded derived category of coherent sheaves, and Perf$(W^{-1}(0))$ is the full triangulated subcategory of perfect complexes. This will be explained in further detail after giving the crepant resolution.

3.2 Construction of the crepant resolution

3.2.1 General definition

Formal statement: $X \to \bar{X} = V/G$ is the crepant resolution given by the Z-Hilbert scheme. We take the Hilbert scheme of 5 points in affine space and then take a Z-version of that.

Recall Z was the group isomorphic to $\mathbb{Z}/5$ generated by a diagonal matrix with fifth roots of unity: (ζ, ζ, ζ^3). $Hilb^5(\mathbb{C}^3)$ consists of Artinian ideals $I \subset \mathbb{C}[v_1, v_2, v_3]$, which means that the Krull dimension of $\mathbb{C}[v_1, v_2, v_3]/I$ is zero (meaning no prime ideal properly contains another) and the quotient should have dimension 5 as a complex vector space. For example we could quotient by $(v_1^2, v_2^2, v_3^2, \ldots)$ so that only 4 monomials (and 1) are left. In particular, the points supported at zero correspond to counting monomials by drawing boxes. (E.g. in 2d can count ideals generated by monomials $x^i y^j$ so that the quotient has dimension 5 by drawing boxes in a plane, where we have powers of x on one axes and powers of y on another, then fix 5 boxes and the ideal I is the complement of those boxes.) This way we get a lot more points at zero – like a blow-up. Then the Z-Hilbert scheme is the component that dominates $\mathbb{C}^3/Z \subset Sym^5(\mathbb{C}^3)^G$ via the map $Hilb^5(\mathbb{C}^3)^G \to Sym^5(\mathbb{C}^3)^G$ (see reference Seidel gives in §2).

However, in this particular case there is an easier way to construct the crepant resolution explicitly. It is done via a toric description.

3.2.2 Toric description in this case

To get $\bar{X}$ use the fan defined given by the single cone $\sigma = N \cap \mathbb{Z}^3_+$ where N is the lattice generated by $\mathbb{Z}^3$ and $\frac{1}{5}(1,1,3)$. Because there is only one cone, we get $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ so $\bar{X}$ is affine. Note $\sigma^\vee$ consists of those points m which have nonnegative inner product with the generators of the cone.

To define the crepant resolution X, first consider the polytope spanned by $\{(1,0,0),(0,1,0),(0,0,1)\}$. The diagram drawn in Seidel's paper is this triangle in the slice given by the $x+y+z=1$ plane, so the picture looks 2-dimensional in his paper. Now triangulate it by adding two more points in the $x+y+z=1$ plane at $\frac{1}{5}(1,1,3)$ and $\frac{1}{5}(2,2,1)$. The cone on this new polytope has 5 cones, which glue together to give X. We get five affine pieces given by the five $\sigma^\vee$ listed in the paper. The gluing maps are given by the chart transformations.

Note that $W^{-1}(0)$ in $V = \mathbb{C}^3$ is invariant under the $Z \cong \mathbb{Z}/5$-action, so descends to a (singular) surface $\bar{H} := W^{-1}(0)/Z \subset \bar{X}$. Under the projection map $\pi : X \to \bar{X}$, the preimage $\pi^{-1}(\bar{H}) =: H$ looks like $\bar{H}$ with a copy of $\mathbb{CP}^2$ along with a copy of the Hirzebruch surface $F_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-3))$.

$$H = \mathbb{CP}^2 \cup F_3 \cup \bar{H}$$

3.3 D^b_{sing} and its relation to matrix factorizations

3.3.1 Definitions

We started with the derived category $D^b(W^{-1}(0))$ of coherent sheaves on $W^{-1}(0) \subset \mathbb{C}^3$. Here are the relevant definitions:

- Coherent sheaf: locally, the sheaf is an $\mathcal{O}_{W^{-1}(0)}$-module with finitely many generators and relations. $\mathcal{O}|_U^m \to \mathcal{O}|_U^n \to \mathcal{F}|_U \to 0$.
- Derived category: objects are chain complexes of objects in the original category. Morphisms: first identify chain homotopic morphisms with each other. Then formally invert morphisms of chain complexes which induce isomorphisms on cohomology. Think of as an analogue of calculus of fractions where we formally invert certain elements.

Then, similar to how going from category to derived category involves formally inverting quasi-isomorphisms, going from $D^b(W^{-1}(0))$ to $D^b(W^{-1}(0))/Perf(W^{-1}(0))$ involves formally inverting morphisms whose cones are perfect complexes. We write:

$$D^b_{sing}(W^{-1}(0)) := D^b(W^{-1}(0))/Perf(W^{-1}(0))$$

Definition 3.3.1. A *perfect* bounded complex is one which is isomorphic to a bounded complex of locally free sheaves in $D^b(W^{-1}(0))$ (i.e. no relations).

Then we take the split closure of the category of singularities (it is already triangulated by construction) to get the triangulated split-closed category $D^\pi_{sing}(W^{-1}(0))$.

Claim 3.3.2. $D^\pi_{sing}(W^{-1}(0))$ is split-generated by the skyscraper sheaf at the origin, denoted $\mathcal{S}_{W^{-1}(0),0}$.

Recall: the skyscraper sheaf at the origin is the sheaf that is zero on all open sets in $W^{-1}(0)$ not containing the origin, and $\mathbb{C}$ on those containing the origin.

Remark 3.3.3. Why is there a "sing" in D^b_{sing} notation? If we are taking it on something smooth, it will be zero. So it sees the singularities in some sense.

3.3.2 Relation to matrix factorizations

An example to keep in mind while defining matrix factorizations in relation to singularities is the variety in $\mathbb{C}^2$ given by the two coordinate axes which is the vanishing $V(xy)$. It is singular because we can factorize the defining polynomial as x times y.

Definition 3.3.4. A *matrix factorization* is a $\mathbb{Z}/2$-graded projective $\mathbb{C}[V]$-module (i.e. summand of a free module) E together with a differential δ_E that factorizes $W \cdot id_E$:

$$\delta_E^2 = W \cdot id_E$$

Such E form a dg-category $MF(W)$ which admits mapping cones, so $H^0(MF(W))$ is triangulated.

W can be thought of as an obstruction to the differential squaring to zero. Now we come to the relation between $MF(W)$ and the category of singularities, due to Orlov.

Theorem 3.3.5. *There is an equivalence of triangulated categories*

$$H^0(MF(W)) \cong D^b_{sing}(W^{-1}(0))$$

sending matrix factorization E to coker$(\delta^1_E : E^1 \to E^0)$.

Because we know the skyscraper sheaf split-generates the singularity category, the natural question is what the corresponding matrix factorization is. The answer is: differential forms $E = \Omega(V)$ with odd/even degree giving a $\mathbb{Z}/2$ grading, and the differential is

$$\delta_E = \iota_\eta - \gamma \wedge \cdot$$

(a degree 1 map because it decreases by 1 in the first term and increases by 1 in the second term, ok for a $\mathbb{Z}/2$-grading). Here $\eta = \sum_i v_i \xi_i$ where v_i is a basis for polynomials on $V = \mathbb{C}^3$, namely for $V^\vee$, and ξ_i is the dual basis of vector fields on V. And $\gamma = \sum_k g_k dv_k \in \Omega^1(V)$ where the g_i are specific polynomials $g_1 = -v_2v_3/3 + v_1^4, g_2 = -v_1v_3/3 + v_2^4, g_3 = -v_1v_2/3 + v_3^4$. The differential squares to $-\gamma(\eta) = W \in \mathbb{C}[V]$. This is where the superpotential reads off the deformation needed to agree with the Fukaya category.

Lemma 3.3.6. *Using the differential δ_E and E above, under Orlov's isomorphism coker(δ^1_E) corresponds with $\mathcal{S}_{W^{-1}(0),0}$.*

We are almost to the statement of finite determinacy and why the A and B-sides are equivalent. To do this, we need to analyze the algebra $\Omega(V)$. We use Koszul duality to go from $\Omega(V)$ above to $\wedge(V)$, and much of the algebra and finite determinacy is then done in $\wedge(V)$.

3.4 The A_∞-str on $\wedge(V)$ in §11: Koszul duality

Let $A = \wedge(V)$ and $B = \mathrm{Hom}_{\mathbb{C}[V]}(\Omega(V), \Omega(V))$. In particular, the forms $\Omega(V)$ form a free Koszul resolution of $\mathbb{C}$ so the cohomology of B is just $\wedge(V)$. There is an A_∞-structure on A that allows us to show finite determinacy. But B is related to the matrix factorizations above. So we want to go between the two.

We can rewrite B as follows:

$$B = \mathrm{Hom}_{\mathbb{C}[V]}(\Omega(V), \Omega(V)) \cong \mathrm{Hom}_{\mathbb{C}}(\wedge(V^\vee), \Omega(V)) \cong \Omega(V) \otimes \wedge(V)$$

The Homological Perturbation Lemma constructs an A_∞-structure $\mathcal{A}$ on $A = \wedge(V)$ and an A_∞-quasi-isomorphism to the dga structure $\mathcal{B}$ on B. Lemma 11.1 tells us that $\mathcal{A}$ agrees with the trivial A_∞-structure on $A = \wedge(V)$ up to order 2 (so zero differential and μ^2 is wedge product). I believe this is what Seidel calls a "deformed version of Koszul duality". The undeformed version is what he does first in §11, where he constructs projection and inclusion

maps p, i between $B = \Omega(V) \otimes \wedge(V)$ and $A = \wedge(V)$ so that on the cohomology level they are isomorphisms (as there is a chain homotopy h between the identity and pi). I believe the given conditions $h^2 = 0, ph = 0, hi = 0$ allow us to use the Homological Perturbation lemma later to construct the quasi-isomorphic structure.

In fact, this A_∞-structure $\mathcal{A}$ agrees with the one constructed in §3,4,5 on $\wedge(V)$. The previous paragraph says it agrees on μ^1, μ^2, and the higher order operations μ^3, μ^5 can be checked directly by calculation. Let's now look at the A_∞-structure on A in §3, 4, 5.

3.5 The A_∞-str on $\wedge(V)$ in §3,4,5: leads to a classification thm

The goal is to show that we only need to compute finitely many A_∞-maps. This is the classification result which is the goal of this subsection (terms and definitions will be explained subsequently):

Proposition 3.5.1 (Proposition 5.2 in Seidel)**.** *Up to equivalence, there is a unique Maurer-Cartan element $\alpha \in \mathfrak{h}^1$ such that $\Phi^1(\alpha_0^3) = -v_1v_2v_3$ and $\Phi^1(\alpha_1^5) = v_1^5 + v_2^5 + v_3^5$.*

Here is an outline of the results leading up to this, which will be explained in more detail:

- We want to find an A_∞-structure on $A = \wedge(V)$ that matches with the A_∞-structure on the Fukaya category on the A-side.
- Start with the algebra $A = \wedge(V)$, which we can match up with the Fukaya category in the product structure as above. We can think of an algebra A as an A_∞-algebra with only one operation, μ^2 which is multiplication.
- Recall Jeff's talk: deformations of A_∞-algebras correspond with Hochschild cohomology $HH(A, A)$. Also mentioned in his talk: a deformation obtained by adding $\{\mu^k\}$ gives an A_∞-structure if and only if μ satisfies the Maurer-Cartan equation with the differential and Gerstenhaber bracket Jeff defined. We want to find the deformation that agrees with the Fukaya category side.
- A classical result says $HH(A, A) \cong \mathbb{C}[[V]] \otimes \wedge(V)$. And the right hand side is easier to work with. We need to note an additional technical point though.
- We need something called filtered pronilpotence (think of as a completion) which will require introducing a formal parameter $\hbar$ into $CC(A, A)$ and $\mathbb{C}[[V]] \otimes \wedge(V)$. The reason for filtered pronilpotence has to do with Lie algebra theory.
- Ultimately the superpotential reads off the A_∞-deformation.
- Finite determinacy follows from using this particular W.

- Note: we are only interested in $\mathbb{Z}/2$-graded categories, so everything done in terms of odd/even parities.

3.5.1 Review from Jeff's talk: $HH(A,A)$ and deformation theory

A is a graded associative algebra over $\mathbb{C}$. The Hochschild complex is

$$CC^d(A,A) = \prod_{i+j-1=d} \mathrm{Hom}^j(A^{\otimes i}, A)$$

This comes equipped with a differential ∂ and a bracket $[,]$ so we get a differential graded Lie algebra. The cohomology with respect to ∂ is $HH(A,A)$.

We now want to deform the usual algebra structure on A. Recall it can be thought of as the A_∞-structure which is zero in all structure maps except μ^2, which is then just the usual product up to sign. This is because the algebra A is associative on the nose. A_∞-deformations correspond to adding maps α^j in each degree to the original A_∞-structure. The resulting maps μ^j should be an A_∞-structure. In other words, $\mu^j = \alpha^j$ in all degrees except 2, where we add a signed product $a_2 a_1$. Note: since A_∞-structure maps are degree $2-d$, the deformation α lives in $CC^1(A,A)$ (note shift down by 1 in Seidel from usual convention where A_∞ is CC^2).

Recall in Jeff's talk: the deformation is A_∞ if and only if $\partial\alpha + \frac{1}{2}[\alpha,\alpha] = 0$. This is the Maurer-Cartan equation. The tangent space of such elements are γ in degree 0 satisfying a certain equation corresponding to gauge transformation (e.g. think about connections).

When $A = \wedge(V)$, there is a classical result that says

$$HH(A,A) \cong \mathbb{C}[[V]] \otimes \wedge(V).$$

It is induced by the Hochschild-Kostant-Rosenberg (HKR) map

$$\Phi^1 : CC(A,A) \to \mathrm{Hom}(T(V), \wedge(V)) \cong \mathbb{C}[[V]] \otimes \wedge(V)$$
$$\Phi^1 : \beta \mapsto (\Phi^1(\beta) : \xi \to \sum_{j=1}^{\infty} \beta^j(\xi, \ldots, \xi))$$

where $T(V)$ is the tensor algebra.

3.5.2 Finding correct deformation in $HH(A,A)$ w/ help of Kontsevich formality

Kontsevich formality says that we can extend this so it is the first term of an L_∞-homomorphism (L_∞ is like A_∞ except we are working the Lie algebras). Now we want a version of this in

the "pronilpotent completion" case. The reason we need filtered pronilpotent comes from Lie group theory. The elements corresponding to A_∞-deformations live in a graded Lie algebra of degree 1, and the tangent space of such deformations live in degree 0. In order to exponentiate elements of the latter using the Baker-Campbell-Hausdorff, we need pronilpotence. So here is the setup to get Kontsevich formality in the completed version.

The main players are Lie algebras $\mathfrak{g}, \mathfrak{h}$ which are the "completed" $\mathbb{C}[[V]] \otimes \wedge(V)$ and $HH(A, A)$. Here are the definitions of each of their respective graded pieces.

$$\mathfrak{g}^d = \prod_{\substack{2i+j+d+1 \equiv 0 \mod 4 \\ 2i+j \geq 3d+3 \\ i \geq d+2}} (\mathrm{Sym}^i(V^\vee) \otimes \bigwedge^j (V))^G \hbar^{\frac{1}{4}(2i+j-3d-3)}$$

$$\mathfrak{h}^d = \prod_{\substack{3i+j-4k=3d+3 \\ k \geq 0, i \geq d+2}} \mathrm{Hom}^j(A^{\otimes i}, A)^G \hbar^k$$

(Note: I removed the k from the notation in the first one, which can be done in the second too.) The Kontsevich formality result here is Lemma 5.1 in Seidel's paper:

Lemma 3.5.2. *There is a filtered L_∞-quasi-isomorphism $\Phi : \mathfrak{h} \to \mathfrak{g}$ whose first term is Φ^1 (where Φ^1 means extend above HKR map $\hbar$-linearly).*

The A_∞-deformations we're interested in live in $\mathfrak{h}$ but by this lemma, we can transfer our problem of finding the deformation to $\mathfrak{g}$. The main point behind finite determinacy in $\mathfrak{h}$ is the corresponding result in $\mathfrak{g}$, which is Lemma 4.1.

Lemma 3.5.3. *Any Maurer-Cartan element $\alpha = (\alpha^0, \alpha^2) \in \mathfrak{g}^1$ such that $\alpha^0 \equiv W \mod F_7\mathbb{C}[[V]]$ is equivalent to $(W, 0)$.*

The filtration $F_\bullet\mathbb{C}[[V]]$ is defined by $F_r\mathbb{C}[[V]]$ consisting of power series where all terms are at least order r.

3.5.2.1 What deformations look like in $\mathfrak{g}$. We need to know what it means to have equivalent deformations on the $\mathfrak{g}$-side. We know that HH^1 classifies deformations to first order, i.e. it's the tangent space to space of deformations. So what classifies deformations in $\mathfrak{g}$? The answer is: look at ideals I where $\mathbb{C}[[V]]/I$ is finite dimensional where I is the *Jacobian ideal* $(\partial_1 W, \partial_2 W, \partial_3 W)$.

Example 3.5.4. Consider the Morse case with the function $x^2 + y^2 + z^2$. Recall that we can always make a Morse function look like this under reparametrization in the domain, up to a constant. And $\mathbb{C}[x, y, z]/(x, y, z)$ has one dimension, corresponding to translation (which is something that does not happen in the domain).

Lemma 3.5.3 above says that we can always reparametrize so that a Maurer-Cartan element that agrees with W up to degree 6, actually agrees with W. This requires come calculation, as done in Seidel's paper. Here is how to see what Maurer-Cartan elements look like, the first part of his calculation.

3.5.2.1.1 Maurer-Cartan equation for $\mathfrak{g}$ First pick $\alpha \in \mathfrak{g}^1$. This can be written as (α^0, α^2) given the grading on $\mathfrak{g}$: look at the possible i, j's in the product. We know that 4 should divide $2i + j + 2$ because $d = 1$ here. Also $i \geq 3$ (so $2i + j \geq 6$ is redundant). If $j = 0$ then i can be any odd number at least 3. And $\bigwedge^j(V)$ is zero for all $j > 3$. So the options are:

- $j = 0$, sum over $i \geq 3$ odd: $\mathbb{C}[v_1, v_2, v_3]_{(i)}$.
- $j = 2$, sum over $i \geq 4$ even: $\mathbb{C}[v_1, v_2, v_3]_{(i)} \otimes \bigwedge^2(V)$.

Hence the two options imply we can write α as (α^0, α^2). Similarly, to talk about the infinitesimal action, we claim we can write $\gamma = (\gamma^1, \gamma^3) \in \mathfrak{g}^0$. We need $4 \mid 2i + j + 1$, $i \geq 2$ (so again $2i + j \geq 3$ redundant). Note that j can only be odd and at most 3. So it could be 1 or 3. If i is odd then $j + 1$ cannot be divisible by 4, so options are:

- $j = 1$, sum over $i \geq 3$ odd: $\mathbb{C}[v_1, v_2, v_3]_{(i)} \otimes V$
- $j = 3$, sum over $i \geq 2$ even: $\mathbb{C}[v_1, v_2, v_3]_{(i)} \otimes \bigwedge^3(V)$

Regarding the action of $\mathrm{GL}(V)$: α is a sum $\alpha^0 + \alpha^2$ and applying bracket (differential is zero in $\mathfrak{g}$) we get zero. The bracket is degree -1 so the function part automatically has bracket zero. We get that the bracket on the two-vector is zero, and α^0 with 2-vector part α^2 is zero. First thing means it defines a Poisson bracket. Second thing means $\{\alpha^0, \cdot\}$ is trivial.

3.6 Conclusion

3.6.1 Finite determinacy

Recall the proposition that is our goal:

Proposition 3.6.1 (Proposition 5.2 in Seidel)**.** *Up to equivalence, there is a unique Maurer-Cartan element $\alpha \in \mathfrak{h}^1$ such that $\Phi^1(\alpha_0^3) = -v_1v_2v_3$ and $\Phi^1(\alpha_1^5) = v_1^5 + v_2^5 + v_3^5$.*

Here the α_j^i are degree j deformations, that input i elements. This proposition follows from the fact that a corresponding statement holds in $\mathfrak{g}$ by working with the Jacobian ideal, and then we can translate the result to $\mathfrak{h}$ by the fact that we have an L_∞-isomorphism with $\mathfrak{h}$ extending the HKR map Φ^1.

So the upshot is, any A_∞-structure satisfying the relations given by the Fukaya category up to order 5 (where the differential is zero, μ^2 is the usual product structure, and μ^3, μ^5 are as in the proposition), is uniquely determined up to isomorphism.

3.6.2 Equivalence of derived categories

A corollary of Orlov's relation between MF and D^b_{sing} is that

$$D^\pi_{sing}(W^{-1}(0)) \cong D^\pi(\mathcal{B})$$

where $\mathcal{B}$ is the A_∞-structure on B as explained earlier. But now we want to quotient by the group action, i.e. we want to know what $D^\pi_{sing}(H)$ is where H is the preimage under the crepant resolution of $\bar{H} = W^{-1}(0)/\mathbb{Z}_5$.

First, going from $W^{-1}(0)$ to $W^{-1}(0)/\mathbb{Z}_5$ means the singularity category becomes isomorphic to $D^\pi(\mathcal{B} \rtimes Z)$. But recall $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic. And recall from earlier with the Fukaya category

$$D^\pi\mathcal{F}(M) \cong D^\pi(\mathcal{A} \rtimes Z)$$

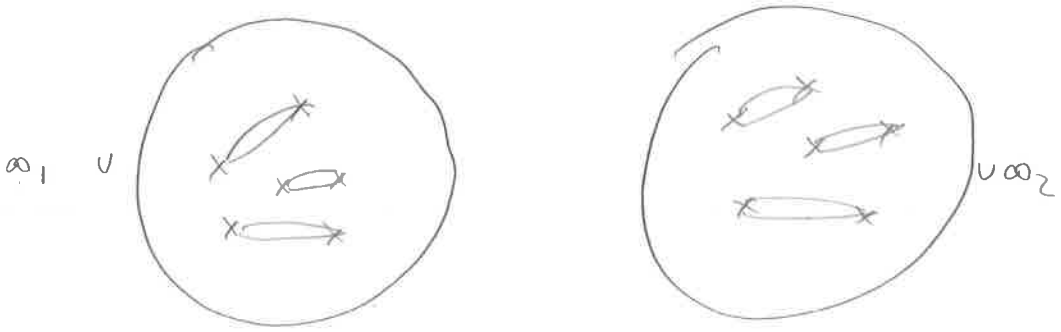
Since we took a crepant resolution, we can go from the category for $\bar{H}$ to that for H and we get the final result:

Theorem 3.6.2.

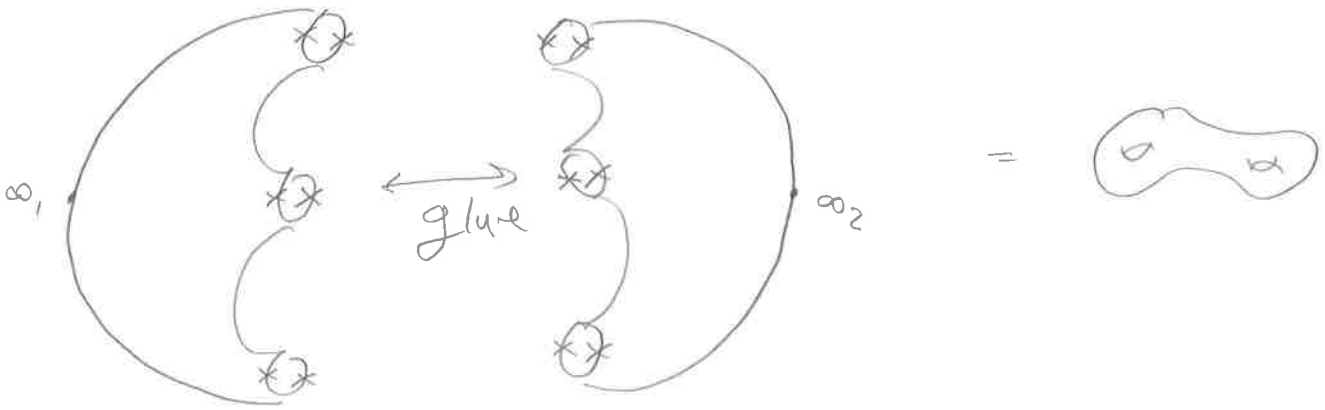
$$D^\pi_{sing}(H) \cong D^\pi(\mathcal{A} \rtimes Z) \cong D^\pi\mathcal{F}(M)$$

4 References

Those Seidel refers to in his paper, as well as Sukhendu Mehrotra's thesis: Triangulated categories of singularities, matrix factorizations and LG-models.

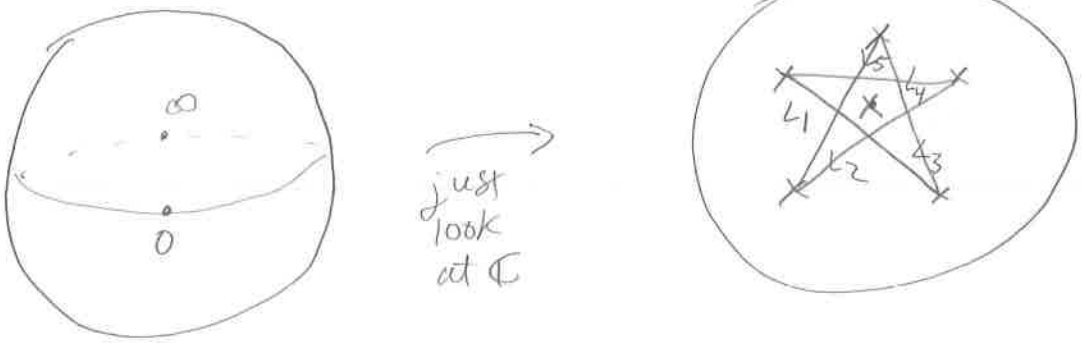


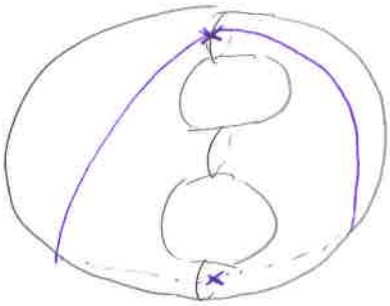
Pull out. cuts





by projection in each factor

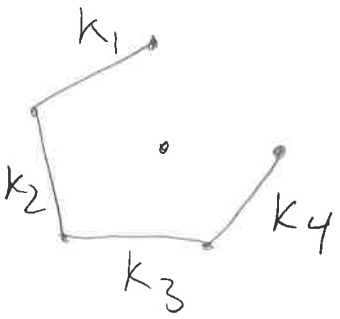


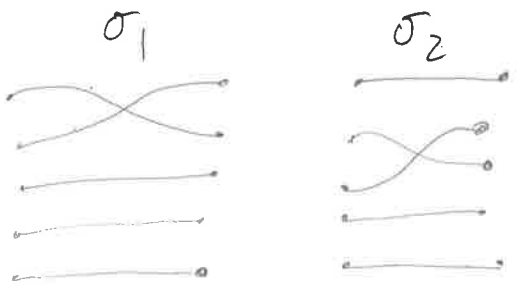


L_1 goes from left, top branch to right, bottom branch

invariant under $\mathbb{Z}/2$-action so projects, in fact projects to L_1

=





& 2 others σ_3, σ_4

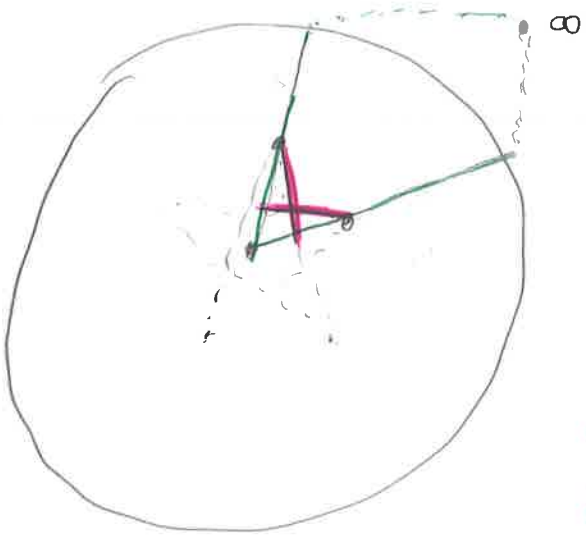
multiplication defined by concatenation.

rels $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| \geq 2$

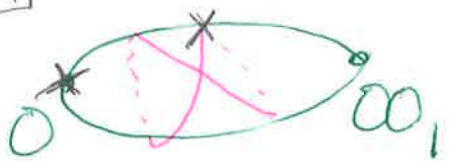
(ie "not adjacent")

③ **Step 1** What does the orbit look like in a "$\frac{1}{5}$-slice"?



green gets identified
pink (w/ ends identified)
is L_1 after quotienting by
$\pi/5$ (i identifying its
images under rotation
by $2\pi/5$).

Pic 1

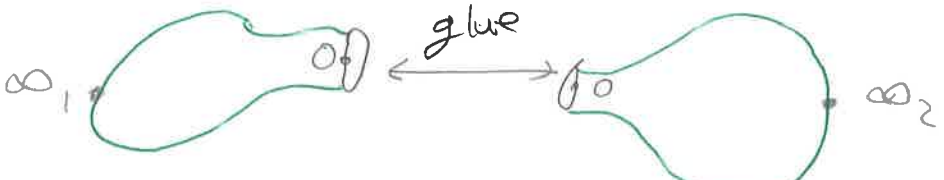


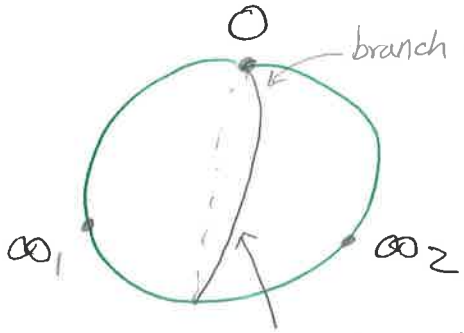
x = branch point

Note: curve goes around 0 and ∞.

Step 2 Glue 2 spheres to get orbifold sphere.

We have 2 copies of Pic 1, glued along branch cut b/w 2 branch points.



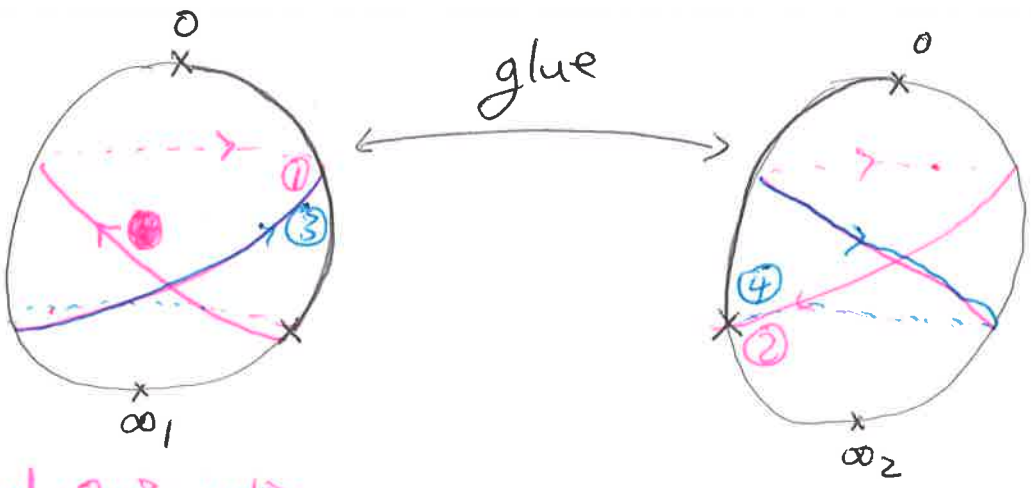


orbifold sphere
$0, \infty_1, \infty_2$ orbifold points

passing thru a branch cut on Pic 1 or Pic 2 corresponds with changing hemispheres in glued picture.

Step 3 What does L_1 (pink curve) glue to on the orbifold sphere?

(just rotated Pic 1 so 0, ∞ are N, S pole)

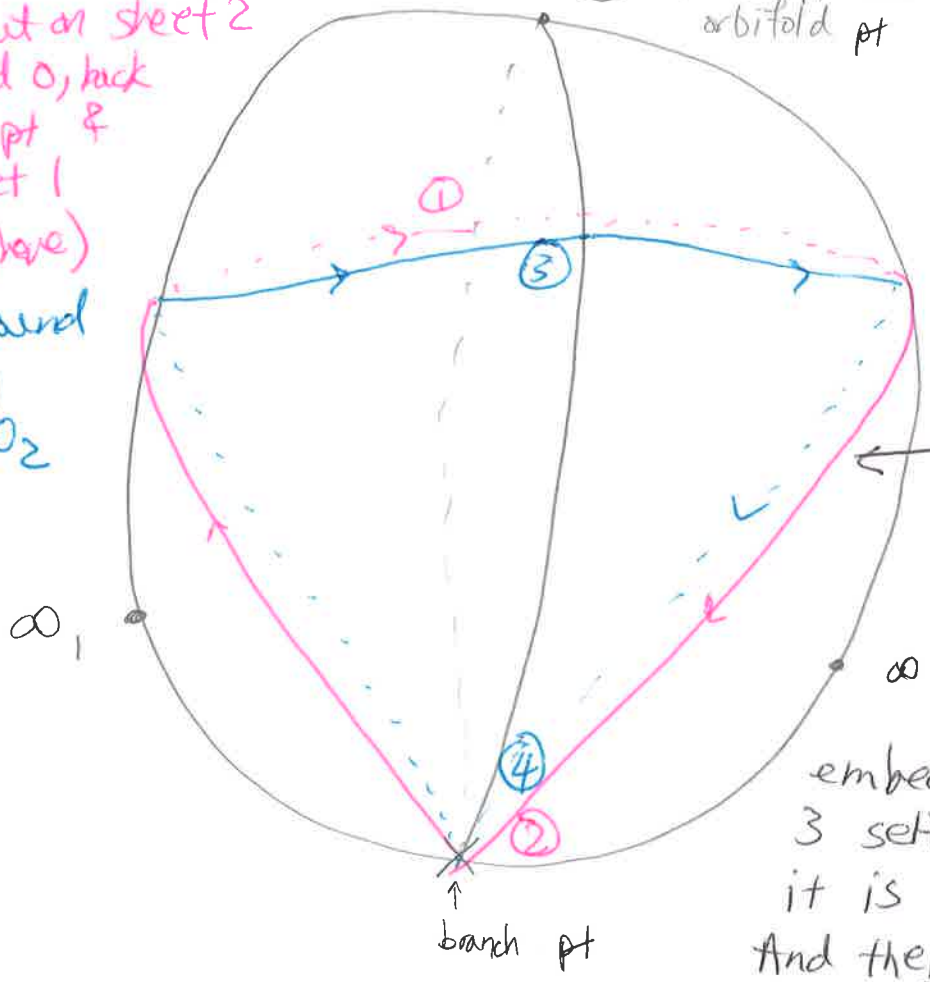


① goes around 0 & onto sheet 2 (ie to right hemisphere)

② comes out on sheet 2 goes around 0, back to branch pt & so to sheet 1 (left hemisphere)

③ goes around ∞_1

④ around ∞_2



great circle = where we glued

UPSHOT

We get a single Lagr which is no longer embedded b/c it has 3 self-∩ pts, but it is still immersed. And there is a Floer theory for immersed curves.