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The quintic 3-fold & its mirror

Simplest Calabi-Yaus = hypersurfaces in toric varieties, esp:

$X =$ smooth hypersurface in $\mathbb{C}P^{n+1}$ defined by a polynomial P of deg. $d = n+2$:
ie. section of $\mathcal{O}_{\mathbb{P}^{n+1}}(d)$

$$\text{Smoothness} \Rightarrow \begin{aligned} NX &\simeq \mathcal{O}_{\mathbb{P}^{n+1}}(d)|_X \\ v &\mapsto \nabla_v P (= dP(v)) \end{aligned}$$

$$\text{so } T\mathbb{P}^{n+1}|_X = TX \oplus NX = TX \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(d)|_X \quad \text{"adjunction"}$$

$$\text{Passing to dual & determinant: } \Omega_{\mathbb{P}^{n+1}}^n|_X \cong \Omega_X^n \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)|_X$$

$$\text{Now: } T_{\mathbb{P}^{n+1}} \otimes \mathbb{C} = \text{Hom}(l, l^\perp) \oplus \text{Hom}(l, l) = \text{Hom}(l, \mathbb{C}^{n+2}) = \text{Hom}(\mathcal{O}(-1)_l, \mathbb{C}^{n+2})$$

$$\Rightarrow T\mathbb{P}^{n+1} \otimes \mathcal{O} \cong \mathcal{O}(1)^{\oplus n+2}$$

$$\& \text{ passing to dual & det: } \Omega_{\mathbb{P}^{n+1}}^n \otimes \mathcal{O} \cong \mathcal{O}(-1)^{\otimes (n+2)} = \mathcal{O}(-(n+2))$$

$$\text{Get: } \mathcal{O}_{\mathbb{P}^{n+1}}(-(n+2))|_X \cong \Omega_X^n \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)|_X$$

$$\Rightarrow \Omega_X^n \cong \mathcal{O} \quad \text{if } d = n+2.$$

Ex. cubic curve in $\mathbb{P}^2 =$ elliptic curve (genus 1, T^2)

quartic surface in $\mathbb{P}^3 =$ K3 surface

quintic in $\mathbb{P}^4 \leftarrow$ the world's most studied CY 3-fold

* Cohomology of the quintic: by Lefschetz hyperplane theorem,

$$\text{inclusion induces } i_*: H_r(X) \xrightarrow{\cong} H_r(\mathbb{C}P^4) \quad \text{for } r < n=3$$

$$\text{so } H_1(X) = 0, \quad H_2(X) = H_2(\mathbb{C}P^4) = \mathbb{Z}$$

$$\rightarrow h^{1,0} = 0 \rightarrow h^{2,0} = 0 \text{ by argument seen before, } h^{1,1} = 1$$

$$\star \chi(X) = e(TX) \cdot [X] = c_3(TX) \cdot [X].$$

$$\text{By working out } c(T\mathbb{P}^4)|_X = c(TX) \cdot c(\mathcal{O}_{\mathbb{P}^4}(5))|_X \quad \text{(adjunction from)}$$

$$c(T\mathbb{P}^4) = c(T\mathbb{P}^4 \otimes \mathcal{O}) = c(\mathcal{O}(1)^{\oplus 5}) = (1+h)^5$$

$$(h = c_1(\mathcal{O}(1)) = \text{generator of } H_2(\mathbb{C}P^4) = \text{P.D. to hyperplane})$$

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$$\rightarrow (1+h_{1X})^5 = 1 + 5h_{1X} + 10h_{1X}^2 + 10h_{1X}^3 = (1+c_1+c_2+c_3)(1+5h_{1X})$$

give $c_1 = 0, c_2 = 10h_{1X}^2, c_3 = -40h_{1X}^3$

so $\chi(X) = -40h^3 \cdot [X] = -40([line] \cap [X]) = -40 \cdot 5 = -200$

conclude: $1 + 1 - \dim H_3(X) + 1 + 1 = -200 \Rightarrow \dim H_3 = 204$
 $H_0 \ H_2 \qquad \qquad \qquad H_4 \ H_6$

Since $h^{3,0} = h^{0,3} = 1$, get $h^{1,2} = h^{2,1} = 101$.

I fact, $\bullet h^{1,1} = 1$ sympl. parameter := area of generator of $H_2(X) (= [line] \in H_2(\mathbb{P}^4))$

$\bullet h^{2,1} = 101$ complex parameters:

equation of quintic: $h^0(\mathcal{O}_{\mathbb{P}^4}(5)) = \binom{9}{5} = 126$

\rightarrow zero sets $\dim \mathbb{P}H^0(\mathcal{O}(5)) = 125$
 minus $\text{Aut}(\mathbb{C}P^4) = \text{PGL}(5, \mathbb{C}) = \dim. 24$) $125 - 24 = 101$

ie. all \mathbb{C} deform^t are still quintics.

* Mirror: start with distinguished family of quintic 3-folds

$$X_\psi = \{ (x_0: \dots : x_4) \in \mathbb{P}^4 / f_\psi = x_0^5 + \dots + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \}$$

let $G = \{ (a_0 \dots a_4) \in (\mathbb{Z}/5)^5 / \sum a_i = 0 \} / (\mathbb{Z}/5) = \{ (a, a, a, a, a) \}$

$(G \cong (\mathbb{Z}/5)^3)$

Then G acts on X_ψ by $(x_j) \mapsto (x_j \zeta^{a_j})$ where $\zeta = e^{2\pi i/5}$

- $\bullet \sum a_j = 0 \pmod 5 \Rightarrow f_\psi$ is G -invariant
- $\bullet x_j$ are homogeneous coords. $\Rightarrow (1,1,1,1,1)$ act trivially

X_ψ is smooth for ψ generic ($\psi^5 \neq 1$), but X_ψ/G singular !!

fixed points = $(x_0: \dots : x_4) \in X_\psi$ s.t. at least two coords. are 0.

This consists of:

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• curve $C_{01} = \{x_0 = x_1 = 0, x_2^5 + x_3^5 + x_4^5 = 0\}$ stabilizer $\mathbb{Z}/5 = \{(a, -a, 0, 0, 0)\}$

$$C_{01}/G \cong \mathbb{P}^1 \quad \left(\begin{array}{l} \text{line } y_2 + y_3 + y_4 = 0 \text{ in } \mathbb{P}^2 \\ y_i = x_i^5 \end{array} \right)$$

one of 10 C_{ij} 's

• point $\in P_{012} = \{x_0 = x_1 = x_2 = 0, x_3^5 + x_4^5 = 0\}$ stabilizer $(\mathbb{Z}/5)^2$

$$5 \text{ pts, } P_{012}/G = \{\text{pt}\}$$

10 such P_{ijk} 's.

\Rightarrow singular locus of $X_\psi/G = 10$ curves $\bar{C}_{ij} = C_{ij}/G (\cong \mathbb{P}^1)$
with $\bar{C}_{ij}, \bar{C}_{jk}, \bar{C}_{ik}$ meeting at a point \bar{P}_{ijk} .

• $\check{X}_\psi :=$ resolution of singularities of (X_ψ/G)

(ie. \check{X}_ψ smooth, $\check{X}_\psi \xrightarrow{\pi} X_\psi/G$, π isom. outside $\pi^{-1}(UC_{ij})$)

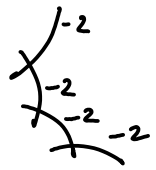
Explicit contrⁿ complicated; can use toric geometry

Can show: This is a crepant resolution (canonical bundle $K_{\check{X}_\psi} = \pi^* K_{X_\psi/G}$)
 \rightarrow CY condition is preserved, \check{X}_ψ is a CY 3-fold.

Ex: along \bar{C}_{ij} (away from \bar{P}_{ijk}), X_ψ/G looks like $(\mathbb{C}^2/\mathbb{Z}_5) \times \mathbb{C}$,
 $(x_1, x_2, x_3) \sim (\xi^a x_1, \xi^{-a} x_2, x_3)$

$$\mathbb{C}^2/\mathbb{Z}_5 \cong \{uv = w^5\} \subset \mathbb{C}^3$$
$$[x_1, x_2] \mapsto [x_1^5, x_2^5, x_1 x_2] \quad A_4\text{-singularity}$$

can resolve by blowing up twice \rightarrow get 4 exc. divisors
(this is crepant!)



Do this for each $\bar{C}_{ij} \rightarrow$ create 40 divisors

Also, resolution at each \bar{P}_{ijk} creates 6 divisors $\times 10 = 60$

$\rightarrow \check{X}_\psi$ contains 100 new divisors besides hyperplane section ...

can in fact show that, indeed, $h^{1,1}(\check{X}_\psi) = 101$

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Can also show $h^{2,1}(\check{X}_\psi) = 1$

(consistent with: we were only able to build a 1-param. family)

So Hodge diamonds match.

quintic

$$\begin{array}{cccc}
 & & 1 & \\
 & 0 & & 0 \\
 & 0 & 1 & 0 \\
 1 & 0 & 1 & 0 & 1 \\
 & 0 & 1 & 0 \\
 & 0 & 0 & \\
 & & 1 &
 \end{array}$$

\check{X}_ψ

$$\begin{array}{cccc}
 & & 1 & \\
 & 0 & & 0 \\
 & 0 & 1 & 0 \\
 1 & 1 & 1 & 1 \\
 & 0 & 1 & 0 \\
 & 0 & 0 & \\
 & & 1 &
 \end{array}$$

We want to see how mirr. symm. predicts the GW invariants N_d ("#rat. curves" n_d) of quintic.

For that, need to understand:

- mirror map between

$$\begin{cases}
 \text{Kähler param. } q = \exp(2\pi i \int_{\text{line}} B + i\omega) \text{ on quintic} \\
 \text{Complex param. } \psi \text{ on mirror } \check{X}_\psi
 \end{cases}$$

(will also give, by differentiating, isom. $H^{1,1}(\text{quintic}) \cong H^{2,1}(\check{X}_\psi)$)

• calcⁿ of Yukawa coupling on $H^{2,1}(\check{X}_\psi)$