1 • <u>Kodaira-Spencer map</u> for complex deform^{ns} of Calabi-Yaus:

$$\begin{array}{ccc}\mathcal{X} & \supset & X \\ \downarrow & & \downarrow \\ S & \ni & 0\end{array}$$ family of deformations of $(X,J) \rightsquigarrow (X,J_t)_{t\in S}$

$c_1(K_X)=0$ (deform. inv^t) and $H^{0,1}=0$ assumed [recall: $K_X := \Omega^{n,0}_X$]

$\Rightarrow K_{X_t} \simeq \mathcal{O}_{X_t}$ holomorphically even after deformation (so all (X,J_t) are Calabi-Yau)

* We've seen: Kodaira-Spencer map $T_0S \to H^1(X,TX)$
  Namely, $J(t)$ deform^n of $J(0)$ is given by $s(t)\in\Omega^{0,1}(X,TX)$
  Fixing a tangent direction $\frac{\partial}{\partial t}\in T_0S \longmapsto \left[\frac{\partial s}{\partial t}\Big|_{t=0}\right]\in H^1(X,TX)$

* Reinterpret Kodaira-Spencer map in terms of holom. vol. form $\Omega_t\in\Omega^{n,0}(X,J_t)$
  $[\Omega_t]\in H^{n,0}_{J_t}(X)\subset H^n(X,\mathbb{C})$. Q^n: how does it depend on t?
  given $\frac{\partial}{\partial t}\in T_0S$, $\frac{\partial}{\partial t}\Omega_t \in \Omega^{n,0}\oplus\Omega^{n-1,1}$ by Griffiths transversality

Recall: Thm: (Griffiths transversality)
$$\alpha_t\in\Omega^{p,q}(X,J_t) \Rightarrow \frac{\partial}{\partial t}\alpha_t\in\Omega^{p,q}+\Omega^{p+1,q-1}+\Omega^{p-1,q+1}$$

[explicit calculation: write $\Omega_t = f_t\, dz_1^{(t)}\wedge\dots\wedge dz_n^{(t)}$ where
$dz_i^{(t)} = dz_i - s_t(dz_i)$ is $(1,0)$ for J_t and differentiate using product rule]

Now: $\frac{\partial\Omega_t}{\partial t}$ is d-closed (since Ω_t d-closed)
$\Rightarrow \left(\frac{\partial\Omega_t}{\partial t}\right)^{(n-1,1)}$ is $\bar\partial$-closed $\Rightarrow \exists\left[\frac{\partial\Omega_t}{\partial t}^{(n-1,1)}\right]\in H^{n-1,1}(X)$

* For fixed Ω_0, this is indep^t of choice of Ω_t. Indeed, could rescale to
  $f(t)\Omega_t$, but then $\frac{\partial}{\partial t}(f(t)\Omega_t) = \underbrace{\frac{\partial f}{\partial t}\Omega_t}_{(n,0)} + \underbrace{f(t)\frac{\partial\Omega_t}{\partial t}}_{(n-1,1)\text{ part scales linearly}}$
  $f(0)=1$

* We have seen $H^{n-1,1}(X) = H^1(X,\Omega^{n-1}_X)\simeq H^1(X,TX)$
  The identification $TX \simeq \Omega^{n-1}_X$, $v\mapsto i_v\Omega$ also depends on choice of Ω;
  the image of $\left[\frac{\partial\Omega_t}{\partial t}\right]^{(n-1,1)}$ in $H^1(X,TX)$ is indep^t of choices and $=\left[\frac{\partial s}{\partial t}\right]$
  ie. this is <u>also</u> the Kodaira-Spencer map.

★ Hence: for $\theta \in H^1(X, TX)$ deform. of complex structure,

$$\theta \cdot \Omega \in H^1(X, \Omega^n_X \otimes TX) \simeq H^1(X, \Omega^{n-1}_X) = H^{n-1,1}(X)$$

and $[\nabla_\theta \Omega^{(n-1,1)}] \in H^{n-1,1}(X)$ are the same thing

Iterating to 3rd order variation ... on a CY-3 fold,

$$\langle \theta_1, \theta_2, \theta_3 \rangle := \int_X \Omega \wedge (\theta_1 . \theta_2 . \theta_3 . \Omega) = \int_X \Omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega$$

(Need 3 derivatives before we can hit a nontrivial (0,3) component...)

<u>Pseudoholomorphic curves:</u> (reference: McDuff-Salamon book)

(X^{2n}, ω) symplectic manifold, J compatible almost-$\mathbb{C}$ structure

($J^2 = -1$, $\omega(\cdot, J\cdot)$ Riem. metric)

(Σ, j) Riemann surface of genus g, $z_1 \dots z_k \in \Sigma$ marked points

moduli space $\mathcal{M}_{g,k} = \{(\Sigma, j, z_1 \dots z_k)\}/_{\text{biholomorphism}}$ ($\dim_{\mathbb{C}} = 3g-3+k$)

Main case for us: (S^2, j), $0, 1, \infty$: $\mathcal{M}_{0,3} = \{pt\}$ so we won't discuss moduli space further.

Def: $u: \Sigma \to X$ is <u>J-holomorphic</u> if $J \circ du = du \circ j$

i.e. $\bar\partial_J u = \frac{1}{2}(du + J\, du\, j) = 0 \in \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes u^* TX)$

Def. $\mathcal{M}_{g,k}(X, J, \beta) = \left\{ (\Sigma, j, z_1, \dots z_k),\ u: \Sigma \to X \ / \ \begin{matrix} u_*[\Sigma] = \beta \\ \bar\partial_J u = 0 \end{matrix} \right\} / \sim$, $\beta \in H_2(X)$

(equivalence relation: $\phi: \Sigma \xrightarrow{\sim} \Sigma'$, $\phi(z_i) = z_i'$, $\begin{tikzcd} \Sigma \arrow[d, "\phi"'] \arrow[r, "u"] & X \\ \Sigma' \arrow[ur, "u'"'] \end{tikzcd}$)

i.e. zero set of a section $\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathrm{Map}(\Sigma, X)_\beta \times \mathcal{M}_{g,k} \arrow[u, bend left, "\bar\partial_J"] \end{tikzcd}$ vector bundle, $\mathcal{E}_u = \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes u^* TX)$

More precisely, look at $W^{k+1,p}$ maps, and $\mathcal{E}_u = W^{k,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes u^* TX)$

$\rightsquigarrow$ Banach bundle over a Banach manifold

The linearized operator $D_{\bar\partial}: W^{k+1,p}(\Sigma, u^* TX) \times T\mathcal{M}_{g,k} \to W^{k,p}(\Sigma, \Omega^{0,1} \otimes u^* TX)$

$$D_{\bar\partial}(v, j') = \bar\partial v + \tfrac{1}{2} \nabla_v J \cdot du \circ j + \tfrac{1}{2} J \cdot du \cdot j'$$

$D_{\bar\partial}$ is Fredholm, of $\text{index}_{\mathbb{R}} = 2d := 2\langle c_1(TX), \beta\rangle + n(2-2g) + \underbrace{(6g-6+2k)}^{\dim \mathcal{M}_{g,k}}$

Q: <u>transversality</u>? i.e. can we get $D_{\bar\partial}$ to be <u>onto</u> at pts of $\mathcal{M}_{g,k}(X,J,\beta)$?

say u is <u>regular</u> if $D_{\bar\partial}$ onto at u.

(if so then $\mathcal{M}_{g,k}(X,J,\beta)$ is smooth of dimension $2d$)

<u>Def</u>: $u: \Sigma \to X$ is <u>simple</u> ("somewhere injective") if $\exists\, z \in \Sigma$ st. $\begin{cases} du(z) \text{ injective} \\ u^{-1}(u(z)) = \{z\} \end{cases}$

otherwise, u factors through a covering $\Sigma \to \Sigma' \to X$

$\mathcal{M}^*_{g,k}(X,J,\beta) = \{\text{simple J-hol. curves}\}$

<u>Thm</u>: $\mathcal{J}^{reg}(X,\beta) = \{ J \in \mathcal{J}(X,\omega) \,/\, \text{every simple J-hol. curve in class } \beta \text{ is regular}\}$ is a Baire subset in $\mathcal{J}(X,\omega)$

For $J \in \mathcal{J}^{reg}(X,\beta)$, $\mathcal{M}^*_{g,k}(X,J,\beta)$ is smooth of real dim. $2d$ and carries a natural orientation.

⚠ in general $\mathcal{M}_{g,k}$ <u>orbifold</u> (Σ with automorphisms)