① Motivation for SYZ conjecture:

Q: ‖ how does one build a mirror $X^\vee$ of a given Calabi-Yau manifold X?

Observe: HMS says $D^b\mathrm{Coh}(X^\vee) \simeq D^\pi \mathcal{F}uk(X)$

In particular, $p \in X^\vee$ point $\leftrightarrow \mathcal{O}_p \in D^b\mathrm{Coh}(X^\vee)$

$\leftrightarrow \mathcal{L}_p \in D^\pi \mathcal{F}uk(X)$.

$X^\vee$ = moduli space of skyscraper sheaves in $D^b\mathrm{Coh}(X^\vee)$

= moduli space of certain objects in $D^\pi\mathcal{F}uk(X)$.

★ What kind of objects?

Recall [Nov 10]: $\mathrm{Ext}^k(\mathcal{O}_p,\mathcal{O}_p) \simeq \Lambda^k V$ ($V \simeq$ tangent space at p)

ie. as graded vector space, $\mathrm{Ext}^*(\mathcal{O}_p,\mathcal{O}_p) \simeq H^*(T^n;\mathbb{C})$

Recall [Oct 22]: in good cases $HF^*(L,L) \simeq H^*(L)$

(though in general, if L bounds holom. discs, only related by a spectral sequence)

⚠ should be with Λ-coefficients, but in good cases can work over a smaller coefficient ring. Since complex side is over $\mathbb{C}$, let's try to use $\mathbb{C}$ as well (set $T = e^{-2\pi}$) and hope for convergence. [Otherwise ... in general recall mirror symm. only holds near LCSL, should have started with a formal family, ie. a scheme over $\Lambda^{\mathbb{C}}$].

★ So if we're optimistic & hope $\mathcal{L}_p$ is actually an honest Lagrangian, then it should be a Lagrangian <u>torus</u>.

In fact there's not enough of those: given $T^n \simeq L \subset X$,

$\nu(L) \simeq T^*L$ and Lagr. deformation of $L \simeq$ graphs of closed 1-forms

Hamiltonian isotopies $\simeq$ graphs of exact 1-forms

$\Rightarrow$ tangent space to "moduli sp. of Lagrangian tori" (⚠) at L is $\cong H^1(L,\mathbb{R})$.

For T^n this is <u>real</u> n-dim$^{\underline{l}}$, half what we want.

② • However: recall <u>twisted</u> Floer homology for (L,∇) [Oct 29]

∇ = flat $U(1)$ conn. on $\underline{\mathbb{C}} \to L$

$(= d+A\ ,\ A \in \Omega^1(L;i\mathbb{R})$ closed) (mod gauge = exact)

∇ affects Floer theory by inserting holonomy factors in disc weights.

$\rightsquigarrow$ actually a more realistic hope is that <u>generic</u> points of $X^\vee$ correspond to isomorphism classes of (L,∇), $L\subset X$ Lagr. torus, ∇ $U(1)$-flat conn.

(some points of $X^\vee$ might still only correspond to objects of the derived Fukaya category).

★ The Strominger-Yau-Zaslow conj. (1996) builds on this and gives a richer geometric picture (get <u>both</u> cx. & sympl. geometry on each of $X, X^\vee$) by picking a preferred representative of the isom. class of (L,∇) (doesn't always exist ⚠).

<u>SYZ conj</u>: $X, X^\vee$ carry dual fibrations by <u>special Lagrangian tori</u>

Ie: $\begin{array}{ccc} T^n \to & X & \\ & \downarrow \pi & \\ & B & \end{array}$, $\begin{array}{ccc} \check{T}^n \to & \check{X} & \\ & \downarrow \pi^\vee & \\ & B & \end{array}$ where $\check{T} = \mathrm{Hom}(\pi_1 T, U(1))$ dual torus

ie. $\check{X} = \{(L,\nabla) \,/\, L$ fiber of π, $\nabla \in \mathrm{hom}(\pi_1 L, U(1))\}$ & vice-versa.

<u>Special Lagrangian</u> := $\omega_{|L} = 0$ <u>and</u> $\mathrm{Im}(\Omega)_{|L} = 0$

holom. volume form

We'll look more into it but there are several warnings:

★ Constructing SLag torus fibrations is difficult & usually impossible. (Joyce, Haase-Zharkov, Gross-Siebert, ...)

<u>general slogan</u>: A LCSL degeneration should give rise to a SLag fibration (the CY metric collapses to B). Still very hard.

③ (Also note: different choice of LCSL degeneration should give a different slag fibration and hence a different mirror).

* SLag fibration will usually have singularities ⇒ dual fibration not well-defined. A related issue = "instanton corrections"

So conjecture as stated mostly applies to tori ... needs to be adjusted in general.

Special Lagrangian submanifolds:

X, ω, J Kähler, g Kähler metric, $\Omega \in \Omega^{n,0}$ holom. volume form

strict Calabi-Yau: g Ricci-flat, $|\Omega|_g =$ const. vs. almost-CY: $|\Omega|_g = \psi \in C^\infty(X, \mathbb{R}_+)$

(point: curvature of Chern connection on $\Omega^{n,0}$ ≡ Ricci form; strict CY ⇔ $\nabla\Omega = 0$)

$\Omega \wedge \bar\Omega = c(n)\, \omega^n$

$\Omega \wedge \bar\Omega = \psi^2 c(n)\, \omega^n$.

— RESTRICT TO STRICT CY CASE FOR BREVITY —

Fact: ‖ $L \subset X$ Lagrangian submfld ⇒ $\Omega_{|L} \in \Omega^n(L, \mathbb{C})$ is of the form
$\Omega_{|L} = e^{i\varphi}\, \psi\, \mathrm{vol}_{g|L}$ with $e^{i\varphi}: L \to S^1$ phase function

(Pf: linear algebra! at a point $p \in L$, ∃ basis of T_pX s.t.
$(T_pX, \omega_p, J_p, T_pL) \simeq (\mathbb{C}^n, \omega_0, J_0, \mathbb{R}^n)$, and $\Omega_p = e^{i\varphi(p)}\psi(p)\, dz_1 \wedge \dots \wedge dz_n$)

Def: ‖ L is special Lagrangian if the phase function is constant.

Then $\int_L \Omega \in e^{i\varphi}\mathbb{R}_+$. Given $[L] \in H_n(X, \mathbb{Z})$, normalize Ω so that $\int_{[L]} \Omega = 1$.

⇝ **Def′:** ‖ L is special Lagrangian iff $\mathrm{Im}\, \Omega_{|L} = 0$.

(and then $\mathrm{Re}\, \Omega_{|L} = \psi \cdot \mathrm{vol}_L$, up to suitable choice of orient^{n} of L)

Rmk 1: in strict CY case, special Lagrangians are calibrated & hence volume-minimizing in their homology class: $\mathrm{Re}\, \Omega_{|\pi} \le \mathrm{vol}_{g|\pi}$ ∀ π n-plane, with equality iff π special Lagrangian. Hence
$[\mathrm{Re}\, \Omega] \cdot [L] = \int_L \mathrm{Re}\, \Omega \le \int_L \mathrm{vol}_g = \mathrm{vol}(L)$ with equality iff S-Lag.

4

Rmk 2: $c_1(TX) = 0 \Rightarrow \exists$ global $\mathbb{Z}$-cover of Lagr. grassmannian of X.

Can describe a graded Lagr. plane as:

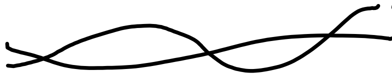
$$\begin{cases} \Pi \subset TX \text{ Lagr. plane} \\ \varphi \in \mathbb{R} \text{ real lift of phase } \arg(\Omega_{|\Pi}) \end{cases}$$

For a general Lagr. $L \subset X$, $e^{i\varphi}: L \to S^1$ may not lift to $\varphi: L \to \mathbb{R}$. Obstruction = homotopy class in $[L, S^1] = H^1(L, \mathbb{Z})$.

Up to factor of 2 this is exactly the Maslov class μ_L.

For L special Lagr., $\mu_L = 0$ automatically ($\Rightarrow$ graded lifts exist, CF^* are $\mathbb{Z}$-graded)

Deformation of special Lagrangians:



$L_t = \exp(tv)$, $v \in C^\infty(NL)$ normal vector field

L

deformation of L

Q^n: when is L_t special Lagrangian? $\quad \varphi_t = \exp(tv): L \to X$

$L_t = \varphi_t(L)$.

- Lagrangians need $\omega_{|L_t} = 0 \ \forall t$, ie. $\varphi_t^* \omega = 0$

1st order condition: $\frac{d}{dt}(\varphi_t^* \omega)_{|t=0} = \mathcal{L}_v \omega = d(\iota_v \omega)$

$\beta = -\iota_v \omega \in \Omega^1(L, \mathbb{R})$ should be closed $\quad d\beta = 0$

- special: need $\operatorname{Im} \Omega_{|L_t} = 0$ ie. $\varphi_t^*(\operatorname{Im}\Omega) = 0$

1st order: $\frac{d}{dt}_{|t=0}(\varphi_t^* \operatorname{Im}\Omega) = \mathcal{L}_v \operatorname{Im}\Omega = d(\iota_v \operatorname{Im}\Omega)$

$\tilde{\beta} = \iota_v \operatorname{Im}\Omega \in \Omega^{n-1}(L, \mathbb{R})$ should also be closed. $d\tilde{\beta} = 0$

$\rightarrow$ Relation between $\beta, \tilde{\beta}$? go back to pointwise linear algebra:

$T_pX \simeq \mathbb{C}^n$, J_0, ω_0, $T_pL = \mathbb{R}^n$, $\Omega_{|p} = \psi\, dz_1 \wedge \dots \wedge dz_n$

$v = \sum a_i \frac{\partial}{\partial y_i} \rightarrow \beta = \sum a_i\, dx_i$

$\tilde{\beta} = \sum a_i \cdot (-1)^{i-1} \psi\, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$

Hence $\tilde{\beta} = \psi * \beta$. (Hodge $*$ for $g_{|L}$)

In strict CY case, $\underline{\tilde{\beta} = *\beta\text{, so } d\beta = d\tilde{\beta} = 0 \Leftrightarrow \beta \text{ harmonic.}}$

5

Prop: 1st order deformations of a special Lagr. submanifold in a strict CY $\cong \mathcal{H}^1(L,\mathbb{R})$.

In almost-CY, 1st order deforms $\simeq \mathcal{H}^1_\psi(L,\mathbb{R}) := \{\beta \in \Omega^1(L,\mathbb{R}) / d\beta = 0, d^*(\psi\beta) = 0\}$

still true that every class in $H^1(L,\mathbb{R}) \ni$ unique ψ-harm. representative.

(Idea: redo Hodge decomp. theorem but with $\Omega^1 \xrightarrow{(d,\psi^{-1}d^*\psi)} \Omega^2 \oplus \Omega^0$
$= (d,d^*) +$ order 0

or... if $\dim n \neq 2$, ψ-harmonic for $g \Leftrightarrow$ harmonic for $\psi^{\frac{2}{n-2}} g$)

Thm: (Mclean / Joyce)

Deformations are unobstructed, ie. moduli space of SLags. is a <u>smooth</u> manifold B with $T_L B \cong \mathcal{H}^1_\psi(L,\mathbb{R})$. $(\simeq H^1(L,\mathbb{R}))$.

Pf: locally near L, deforms $\overset{\exp}{\Longleftrightarrow}$ normal vector fields. Consider the Banach bundle $\mathcal{E}$ over $\mathcal{U} \subset W^{k,p}(L,NL)$ with fiber at v $W^{k-1,p}(L,\Lambda^2T^*L) \oplus W^{k-1,p}(L,\Lambda^nT^*L)$, and the section $s(v) = (\exp(v)^*\omega, \exp(v)^*\operatorname{Im}\Omega)$; Then $B = s^{-1}(0)$.
ω, $\operatorname{Im}\Omega$ closed $\Rightarrow$ $s(v)$ always takes values in closed forms, and looking at Lie derivatives, since $s(0)=0$, <u>exact</u> forms.
$\mathcal{F} \subset \mathcal{E}$ Banach subbundle of exact forms, then s is a Fredholm section of $\mathcal{F}$, and $ds(0)\circ(\omega^\#)^{-1} : \beta \mapsto (-d\beta, d(\psi *\beta))$ is onto
$\begin{pmatrix} \omega^\# : NL \xrightarrow{\sim} T^*L \\ v \mapsto -i_v\omega \end{pmatrix}$ $\Rightarrow s^{-1}(0)$ smooth. △

- We have 2 canonical isoms. $T_LB \xrightarrow{\sim} H^1(L,\mathbb{R})$, $v \mapsto [-i_v\omega]$ "symplectic" and $T_LB \xrightarrow{\sim} H^{n-1}(L,\mathbb{R})$, $v \mapsto [i_v \operatorname{Im}\Omega]$ "complex"

Def: An <u>affine structure</u> on a mfld N^n is a set of coord. charts with transition functions in $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$

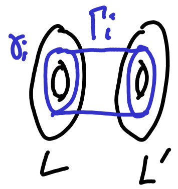
Corollary: B carries two natural affine structures

Slogan: "Mirror symmetry = interchange of the affine structures"

* Case of interest to us: special Lagr. <u>tori</u> $\rightsquigarrow$ then $\dim H^1 = n$.
  Usual harmonic 1-forms for flat metric on $L = T^n$ have no zeroes (pointwise form basis of $T^*L = NL$); standing assumption: this holds for ψ-harmonic 1-forms wrt $g_{|L}$ too.

  Then a nbd of L is fibred by special Lagr. deformations of L, ie. locally
  $$\begin{array}{ccc} T^n & \longrightarrow & U \subset X \\ & & \downarrow \pi \\ & & B \end{array}$$
  Slag fibration

* <u>Local affine coordinates:</u> pick basis $\gamma_1 \cdots \gamma_n$ of $H_1(L, \mathbb{Z})$



$\rightsquigarrow x_i = \int_{\Gamma_i} \omega$ affine coordinates on B for sympl. affine str. (= flux for deform^n of L).

Dually, $\gamma_1^* \cdots \gamma_n^*$ basis of $H_{n-1}(L, \mathbb{Z}) \rightsquigarrow$
$$x_i^* = \int_{\Gamma_i^*} \mathrm{Im}\, \Omega$$ affine coords for complex affine structure

This only works <u>locally</u>: globally there's monodromy. The linear part $\in GL(n, \mathbb{Z})$ is given by monodromy of the Slag family: $\pi_1(B, *) \to \begin{matrix} GL(H^1(L, \mathbb{Z})) \\ GL(H^{n-1}(L, \mathbb{Z})) \end{matrix}$
(Poincaré <u>dual</u> of each other $\Rightarrow$ get <u>transpose</u> monodromies)

* <u>Prototype construction of mirror pair:</u>
  B affine mfld $\rightsquigarrow$ lattice $\Lambda \subset TB$ ($\Leftrightarrow$ integer vectors in affine charts)
  Then TB/Λ torus bundle / B carries a natural cx. structure (J(base) = fiber ...)
  T^*B/Λ^* carries a natural sympl. structure

  MS exchanges complex mfld $TB/\Lambda \longleftrightarrow$ Sympl. mfld T^*B/Λ^*.

  In our case, B carries 2 affine structures with mutually dual monodromies:
  $$\begin{array}{ccc} TB & \xrightarrow{\sim} & T^*B \\ \text{cx.}\ \wr & & \wr\ \text{sympl.} \\ H^{n-1}(L, \mathbb{R}) & \underset{\text{P.D.}}{\simeq} & H_1(L, \mathbb{R}) \\ \cup & & \cup \\ \Lambda_c = H^{n-1}(L, \mathbb{Z}) & \simeq & H_1(L, \mathbb{Z}) = \Lambda_s^* \end{array}$$
  ie. $\underset{\text{cx. geom.}}{TB/\Lambda_c} \simeq \underset{\text{sympl. geom.}}{T^*B/\Lambda_s^*}$

  and dually for the mirror geometry.

* Let's construct the candidate mirror more explicitly: [see also Hitchin]

Let $M = \{ (L, \nabla) \,/\, L \text{ special Lagr.}, \nabla \text{ flat } U(1) \text{ conn./gauge} \}$

(ie. $\nabla = d + A$, $A \in \Omega^1(L, i\mathbb{R})$, $dA = 0$, mod exact forms)

$$T_{(L,\nabla)} M \cong \{ (v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L, i\mathbb{R}) \,/\, -\iota_v \omega \in \mathcal{H}^1_\psi(L, \mathbb{R}),\ d\alpha = 0 \} / 0 \times \operatorname{Im} d$$

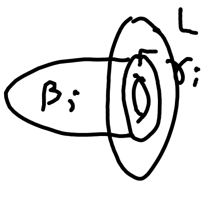
$$\cong \{ (v, i\alpha) \in \text{———} \,''\, \text{———} \,/\, -\iota_v \omega + i\alpha \in \mathcal{H}^1_\psi(L, \mathbb{C}) \}$$

Complex vector space $\Rightarrow$ M carries a natural <u>almost-$\mathbb{C}$ structure</u> $J^\vee$.

<u>Prop</u>: $J^\vee$ is <u>integrable</u>.

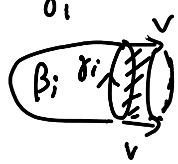
<u>Pf</u>: enough to give local holom. coordinates.

$\gamma_1 \dots \gamma_n$ basis of $H_1(L, \mathbb{Z})$; assume each $\gamma_i = \partial \beta_i$, $\beta_i \in H_2(X, L)$



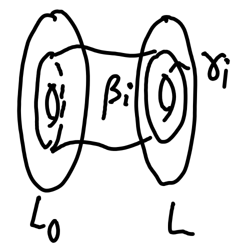
Then set $z_i(L, \nabla) := \exp(-\int_{\beta_i} \omega)\ \mathrm{hol}_\nabla(\gamma_i) \in \mathbb{C}^*$

$$\rightsquigarrow d\log z_i(v, i\alpha) = -\int_{\gamma_i} \iota_v \omega + i \int_{\gamma_i} \alpha_i = \langle \underbrace{[-\iota_v \omega + i\alpha_i]}_{H^1(L,\mathbb{C})}, [\gamma_i] \rangle$$



basis of $T^* M^{1,0}$ ✓

If such β_i don't exist, do the same with



⚠ EVERYTHING UP TO FACTORS OF 2π

- <u>Holom. $(n,0)$-form</u>: $\check{\Omega}((v_1, i\alpha_1) \dots (v_n, i\alpha_n)) = \int_L (-\iota_{v_1} \omega + i\alpha_1) \wedge \dots \wedge (-\iota_{v_n} \omega + i\alpha_n)$

  (if take γ_i "standard" basis above, then in above coords. $\check{\Omega} = \prod d\log z_i$)

- <u>Kähler form</u>: $\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) := \int_L \alpha_2 \wedge \iota_{v_1} \operatorname{Im} \Omega - \alpha_1 \wedge \iota_{v_2} \operatorname{Im} \Omega$

  [recall we've normalized $\int_L \Omega = 1$]

<u>Prop</u>: $\omega^\vee$ is a Kähler form compatible with $J^\vee$

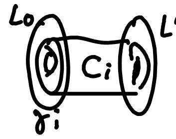
8

Pf: pick $[\gamma_i]$ basis of $H_{n-1}(L,\mathbb{Z})$, $[e_i]$ basis of H_1 s.t. $e_i \cdot \gamma_j = \delta_{ij}$.

Then $\forall\, a \in H^1(L)$, $b \in H^{n-1}(L)$, $\langle a \cup b, [L] \rangle = \sum_i \langle a, e_i \rangle \langle b, \gamma_i \rangle$ (*)

(think: $e_i = i^{th}$ coord. axis, $\gamma_i = i^{th}$ hyperplane)

let $p_i = \int_{C_i} \operatorname{Im} \Omega$,



(affine coords for $\mathbb{C}$ affine structure)

$\theta_i = \int_{e_i} A$ (i.e. $\operatorname{hol}_{e_i}(\nabla) = e^{i\theta_i}$)

Then $dp_i : (v, \alpha) \mapsto \int_{\gamma_i} \iota_v \operatorname{Im} \Omega = \langle [\iota_v \operatorname{Im} \Omega], \gamma_i \rangle$

$d\theta_i : (v, \alpha) \longmapsto \int_{e_i} \alpha = \langle [\alpha], e_i \rangle$

and (*) $\Rightarrow$ $\omega^\vee = \sum dp_i \wedge d\theta_i$ ($\Rightarrow$ closed).

Now: $\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) = \int_L \psi \left(\langle \alpha_1, \iota_{v_2} \omega \rangle_g - \langle \alpha_2, \iota_{v_1} \omega \rangle_g \right)$

$\Rightarrow$ $\omega^\vee((v_1, \alpha_1), J^\vee(v_2, \alpha_2)) = \int_L \psi \left(\langle \alpha_1, \alpha_2 \rangle_g + \langle \iota_{v_1} \omega, \iota_{v_2} \omega \rangle_g \right)$

clearly a Riemannian metric ✓ ∎