(1) **Derived categories:** slogan: consider complexes up to homotopy.

★ enlarging a category to include complexes of objects makes it
- algebraically better behaved (e.g.: der. cat is triangulated)
- less sensitive to initial data (can restrict to nice subset of objects)
  (e.g.: on a smooth alg. var., coherent sheaves have a finite resolution by vector bundles, so can start with vector bundles instead of coherent sheaves ...)

(more important for Fukaya categories: allow immersed Lagrangians? ...)

★ even if we know how to define general objects, it's usually easier to replace them by complexes of better-behaved objects.

E.g. $\mathcal{O}_D$, $D = s^{-1}(0)$, $s \in H^0(\mathcal{L})$ $\longleftrightarrow$ resolve by complex $\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X$

or Koszul resolution used last time to compute Ext's for $\mathcal{O}_p$

Another example: intersection theory works better with complexes of nice objects

$D_1, D_2 \subset X$ smooth cx. surface defined by sections s_1, s_2 of $\mathcal{L}_1, \mathcal{L}_2$

If $D_1 \pitchfork D_2$ then $\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2} = \mathcal{O}_{D_1 \cap D_2}$ contains "the right information"

can also resolve by complex $\mathcal{L}_1^{-1}{}_{|D_2} \xrightarrow{s_1|D_2} \mathcal{O}_{D_2}$ $\quad$ (Coker $= \mathcal{O}_{D_1 \cap D_2}$)

($=$ apply $-\otimes \mathcal{O}_{D_2}$ to $\mathcal{L}_1^{-1} \xrightarrow{s_1} \mathcal{O}_X$)

But in non-transverse case, e.g. $D_1 = D_2 = D$, $\mathcal{O}_D \otimes \mathcal{O}_D = \mathcal{O}_D$ looks different?

Point: should instead work at level of complexes and apply $-\otimes \mathcal{O}_D$ to the resolution $\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X$ of $\mathcal{O}_D$, to get $\mathcal{L}^{-1}{}_{|D} \xrightarrow{s_{|D} = 0} \mathcal{O}_D$

Cokernel of $\mathcal{L}^{-1}{}_{|D} \xrightarrow{0} \mathcal{O}_D$ is still $\mathcal{O}_D$, but now there's also a kernel, which is the information we lost...

[information was lost because $-\otimes \mathcal{O}_D$ is only right exact, so $0 \to \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_D \to 0$ only yields $\mathcal{L}^{-1}{}_{|D} \xrightarrow{0} \mathcal{O}_D \to \mathcal{O}_D \to 0$. By contrast $-\otimes \mathcal{O}_D$ is exact on vector bundles]

★ However: a same object may have many different resolutions...
when do we want to treat 2 complexes as isomorphic?
Looking at resolutions, it's tempting to think $H^*(\text{complex})$ is what we want, but this is much too coarse – loses important information.

② E.g. <u>Whitehead</u>: X, Y simplicial complexes, simply connected:

then $X \underset{h.e.}{\sim} Y$ iff. $\exists$ simplicial complex Z & maps $X \nearrow Z \nwarrow Y$

s.t. chain maps

$$\begin{tikzcd} & C^*(Z) \arrow[dl] \arrow[dr] & \\ C^*(X) & & C^*(Y) \end{tikzcd}$$

are isomo. on cohomology.

(e.g., if $f: X \to Y$ h.e., Z = mapping cylinder)

(whereas $H_*(X) \simeq H_*(Y)$ doesn't imply much, e.g. Massey products...)

(pass through Z = need to subdivide X / Y so homotopy equiv^{ce} between them can be approximated by a simplicial map)

<u>Def</u>: $C_\bullet \xrightarrow{f} D_\bullet$ chain map (i.e.

$$\begin{tikzcd} \cdots C_i \arrow[r, "\partial"] \arrow[d, "f"] & C_{i+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_{i+2} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots D_i \arrow[r, "\partial"] & D_{i+1} \arrow[r, "\partial"] & D_{i+2} \arrow[r] & \cdots \end{tikzcd}$$

)

is a <u>quasiisomorphism</u> if the induced maps on cohomology are isomorphisms

This is stronger than $H^*(C_\bullet) \simeq H^*(D_\bullet)$

<u>Ex</u>: $\mathbb{C}[x,y]^2 \xrightarrow[(x,y)]{} \mathbb{C}[x,y]$ and $\mathbb{C}[x,y] \xrightarrow[0]{} \mathbb{C}$

not quasiisomorphic as complexes of $\mathbb{C}[x,y]$-modules even though same H^*

<u>Ex</u>: $\{\mathcal{L}^{-1} \xrightarrow[s]{} \mathcal{O}_X\}$ and $\mathcal{O}_D$ are quasiisomorphic, q.isom = cokernel map

(similarly with other resolutions of coherent sheaves).

<u>Def^{ns}</u>: • an <u>additive category</u> := • $\mathrm{Hom}(A,B)$ abelian groups
• Composition is distributive (bilinear)
• $\exists$ direct sums of objects $A \oplus B$
• $\exists$ zero object 0 $(\mathrm{hom}(0,A) = \mathrm{hom}(A,0) = 0)$

• <u>abelian category</u> = additive cat. s.t. all morphisms have ker & coker

[everything defined by univ. properties, e.g. kernel of $A \xrightarrow{f} B$ is $K \to A$ st. $g: C \to A$ factors (uniquely) through K iff $f \circ g = 0$.
In actual examples, ker/coker are always "usual" ones].

$\rightsquigarrow$ in an abelian cat. we have notions of - exact sequence
- cohomology of a complex.

(3)

Def: $\mathcal{A}$ abelian category $\rightsquigarrow$ the <u>bounded derived cat.</u> $D^b(\mathcal{A})$:
* objects = bounded (ie., finite length) chain complexes in $\mathcal{A}$
* morphisms = chain maps up to homotopy, localizing wrt quasi-isoms.

• <u>homotopy</u>:

$$\begin{tikzcd}
\cdots A_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & A_i \arrow[r, "d_i"] \arrow[dl, "h_i"'] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[dl, "h_{i+1}"'] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] & {} \\
\cdots B_{i-1} \arrow[r, "d'_{i-1}"'] & B_i \arrow[r, "d'_i"'] & B_{i+1} \arrow[r, "d'_{i+1}"'] & {}
\end{tikzcd}$$

f, g are homotopic ($f \sim g$) if $\exists h: A \to B[-1]$ st. $f - g = d_B h + h d_A$.
Then look at chain maps$/\sim$

• <u>Equivalently</u>: bounded complexes form a differential graded category
morphisms = "prehoms of complexes" $\underline{\mathrm{Hom}}^k(A_\bullet, B_\bullet) = \bigoplus_i \mathrm{Hom}_{\mathcal{A}}(A_i, B_{i+k})$
differential = $f \in \underline{\mathrm{Hom}}^k(A_\bullet, B_\bullet) \Rightarrow \delta(f) = d_B f + (-1)^{k+1} f d_A$.
Then chain maps = $\ker(\delta : \underline{\mathrm{Hom}}^0 \to \underline{\mathrm{Hom}}^1)$
nullhomotopic = $\mathrm{Im}(\delta : \underline{\mathrm{Hom}}^{-1} \to \underline{\mathrm{Hom}}^0)$
$\Rightarrow$ we want to consider $H^0 \underline{\mathrm{Hom}}(A, B)$.

• <u>Localization</u> wrt quasi isoms := formally invert quasi-isos, ie. add extra morphisms s^{-1} whenever s is a quasi-iso.
In other terms, $\mathrm{Hom}_{D^b(\mathcal{A})}(A_\bullet, B_\bullet) = \{ A \xleftarrow[\text{quasi iso}]{s} A' \xrightarrow[\text{chain map}]{f} B \}/\sim$

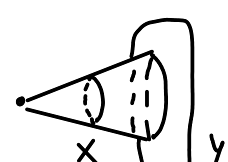
[<u>NB</u>: can skip quotienting by homotopies, because homotopy equivalences are quasi-isomorphisms, but keeping it makes things more explicit].

• <u>Similarly: $D^+(\mathcal{A})$, $D^-(\mathcal{A})$</u> (complexes bounded below, bounded above).

<u>Cones and triangles</u>:

• in category of top. spaces (or simplicial complexes etc.), $\nexists$ ker & coker !!
(unless map is a fibration or an inclusion). However, <u>mapping cone</u> acts as both simultaneously:

$f: X \to Y \rightsquigarrow C_f := (X \times [0,1]) \amalg Y \Big/ \begin{matrix} (x,0) \sim (x',0) \\ (x,1) \sim f(x) \end{matrix}$



④

There are natural maps $Y \to C_f$ (inclusion) and $C_f \to \Sigma X$ (collapse Y) [Σ: suspension]

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X \to \cdots$$

with compositions null homotopic, giving long exact sequence

$$H_i(X) \to H_i(Y) \to H_i(C_f) \to \underset{H_{i-1}(X)}{\underset{\shortparallel}{H_i(\Sigma X)}} \to \underset{H_{i-1}(Y)}{\underset{\shortparallel}{H_i(\Sigma Y)}} \to \cdots$$

if X, Y simplicial complexes

$\rightsquigarrow$ C_f simplicial complex with i-cells $= \begin{cases} \text{cones on } (i-1)\text{ cells of } X \\ i\text{-cells of } Y \end{cases}$, $\partial = \begin{pmatrix} \partial_X & 0 \\ f & \partial_Y \end{pmatrix}$

<u>By analogy:</u> $f : A^\bullet \to B^\bullet$ chain map b/w complexes

$$\rightsquigarrow C_f := A[1] \oplus B, \quad d = \begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$$

ie $C_f^i = A^{i+1} \oplus B$

<u>E.g:</u> if A, B are single objects, $\mathrm{Cone}(f : A \to B)$ is just $\{A \xrightarrow{f} B\}$

We have natural chain maps $B^\bullet \xrightarrow{i} C_f^\bullet$ (inclusion of B as subcomplex)

$C_f^\bullet \xrightarrow{q} A^\bullet[1]$ (quotient complex)

(Can check $A^\bullet[1]$ is quasi-isomorphic to mapping cone of $i : B^\bullet \to C_f^\bullet$)

Thus, in derived category we don't have kernels & cokernels, but we have <u>exact triangles</u>

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd} \quad \text{or} \quad A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

(with corresp. long exact seqs. in cohomology of complexes

$$H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots)$$

$\rightsquigarrow$ $D^b(\mathcal{A})$ is a <u>triangulated category</u>, namely additive cat. with a shift functor $T = [1]$ and a set of "distinguished triangles" satisfying various axioms, among which:

- $\forall X \in \mathrm{ob}$, $\begin{tikzcd} X \arrow[rr, "id"] & & X \arrow[dl] \\ & 0 \arrow[ul] & \end{tikzcd}$ is a distinguished triangle

- $\forall f : X \to Y$, $\exists$ dist. triangle $\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & C \arrow[ul, "{[1]}"] & \end{tikzcd}$