1. **Recall:** we've constructed product operations satisfying A_∞-relations

$$m_k : CF(L_0, L_1) \otimes \dots \otimes CF(L_{k-1}, L_k) \to CF(L_0, L_k)[2-k]$$

counting J-holomorphic disks with boundary on given Lagrangians [under assumptions of transversality & absence of disk bubbling].

We now introduce a version of the Fukaya category more relevant to HMS.

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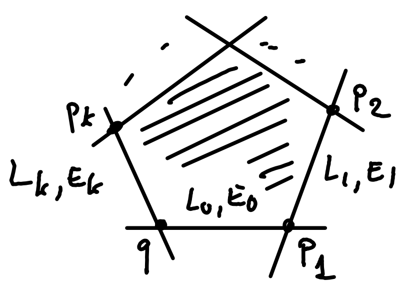
* **Twisted coefficients:**

L_i Lagrangians are equipped with $(E_i, \nabla_i) \to L_i$ vector bundles w/ flat connections (think of: $\mathbb{C}$ vect. bundle w/ flat unitary conn., but could generalize to Novikov).

Define $CF((L_0, E_0, \nabla_0), (L_1, E_1, \nabla_1)) = \bigoplus_{p \in L_0 \cap L_1} Hom((E_0)_p, (E_1)_p) \otimes \Lambda$

Then given $p_1 \cdots p_k$ $(p_i \in L_{i-1} \cap L_i)$ and $w_i \in Hom((E_{i-1})_{p_i}, (E_i)_{p_i})$,

let $m_k(w_k \cdots w_1) := \sum_{\substack{q \in L_0 \cap L_k \\ [u]}} (\# \mathcal{M}(p_1 \cdots p_k, q, [u], J))\, T^{\omega([u])} \underbrace{\mathcal{P}_{[\partial u]}(w_k \cdots w_1)}_{\in Hom((E_0)_q, (E_k)_q)}$



parallel transport along ∂u from q to p_1 using ∇_0 gives $\gamma_0 \in Hom((E_0)_q, (E_0)_{p_1})$

p_i $\quad p_{i+1}$ $\quad \nabla_i$ $\quad \gamma_i \in Hom((E_i)_{p_i}, (E_i)_{p_{i+1}})$

p_k $\quad q$ $\quad \nabla_k$

Flatness of $\nabla_i \Rightarrow$ these depend only on homotopy class of u.

$$\rightsquigarrow \mathcal{P}_{[\partial u]}(w_k \cdots w_1) := \gamma_k \circ w_k \circ \gamma_{k-1} \circ \dots \circ \gamma_1 \circ w_1 \circ \gamma_0 \in Hom((E_0)_q, (E_k)_q)$$

Esp. important to us: E_i = top. trivial line bundle $\mathbb{C} \times L_i$

∇_i = flat $U(1)$ connection $\nabla_i = d + iA_i$, A_i closed 1-form

Then $CF = \bigoplus_{p \in L_0 \cap L_1} \Lambda_{\mathbb{C}}\, p$, using generators $(p, w = Id : E_{0_p} \xrightarrow{\sim} E_{1_p})$

②

$\Rightarrow m_k$ counts discs with weights $T^{\omega(u)}\,\mathrm{hol}(\partial u)$

where $\mathrm{hol}(\partial u) \in U(1)$ = holonomy for parallel transport around loop ∂u, defined using identification at corners $= \exp(i \sum_{j=0}^{k} \int_{(\partial u)_j} A_j)$

- First iteration of Fukaya category (as an A_∞-precat.)
  - objects = $\mathcal{L} = (L, E, \nabla)$,

    L compact spin Lagrangian ($\mathbb{Z}$-graded version: $\mu_L = 0$, + grading data) s.t. L doesn't bound holom. discs.

    (E, ∇) flat hermitian vector bundle
  - for $\mathcal{L}_0 \pitchfork \mathcal{L}_1$, $\hom(\mathcal{L}_0, \mathcal{L}_1) := CF^*$ Floer complex
  - for transverse sequence, m_k = operations on Floer complex.

★ "Convergent power series" Floer homology:

We've recorded holom. disks with weights $T^{\omega(u)}$.

Gromov compactness $\Rightarrow$ Σ may be infinite but well-def! in Novikov ring Λ

Physicists would actually write $e^{-2\pi\omega(u)} \in \mathbb{R}$ and hope for convergence.

Working over Λ, from a physicist's perspective, amounts to considering a family of symplectic forms $(M, \omega_t = t\omega)$ $(\Leftrightarrow T = e^{-2\pi t})$ near the large volume limit $(t \to \infty)$ and computing Floer homologies for all ω_t simultaneously (for t large, if radius of convergence is nonzero; or purely as a formal family near large vol. limit).

Beware: even when it is defined, convergent power series HF^* need not be a Hamiltonian isotopy invariant.

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2 major outstanding issues: • L_0 not transverse to L_1? • L bounds discs?

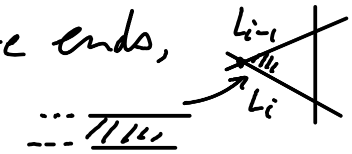
1) what to do if L_0, L_1 not transverse? in particular, $CF^*(L,L)$?

Various approaches in literature:

a) pick a Hamiltonian perturbation to make them transverse.

(ie., define $CF^*(L_0, L_1)$ to be generated by $L_0 \cap \varphi_H(L_1)$ where $H = H(L_0, L_1)$, and perturb all holom. curve equations by suitable

③ Hamiltonian terms — in particular, in strip like ends, want $H \to H(L_{i-1}, L_i)$

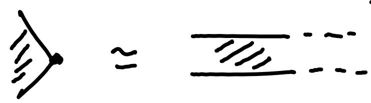


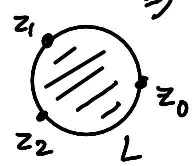
See e.g. Seidel's book.

<u>Main issues</u>: (at chain level! Floer homology easier...)

- need to fix consistent choices of perturbation data.
  (need a procedure which associates to each pair (L, L') a Hamiltonian $H(L,L')$, and to each sequence $(L_0 \dots L_k)$, perturbation data for $(k+1)$-marked holom. discs s.t. converges to $H(L_{i-1}, L_i)$ in each strip-like end)
  + show different choices yield equivalent categories
- no canonical strict unit $1 \in CF^*(L,L)$. (only a homology unit)

b) <u>"Morse-Bott" Floer homology</u> (e.g. FOOO)

- $CF^*(L,L) := C_*(L;\Lambda)$ "singular chains" on L (in a suitable sense...)
  Operations m_k: instead of a strip-like end 
  put a boundary marked point  and require $u(z) \in$ chain.

  <u>E.g.</u>: product m_2 considers  $ev_i : \bar{\mathcal{M}}_{0,3}(X, L; J, \beta) \to L$

  $$m_2(C_2, C_1) = \sum_{\beta \in \pi_2(X,L)} ev_{0*}\left([\bar{\mathcal{M}}_{0,3}(X,L;J,\beta)] \cap ev_1^* C_1 \cap ev_2^* C_2\right) T^{\omega(\beta)}$$

  contribution of constant discs $\equiv$ intersection product $C_*(L)$

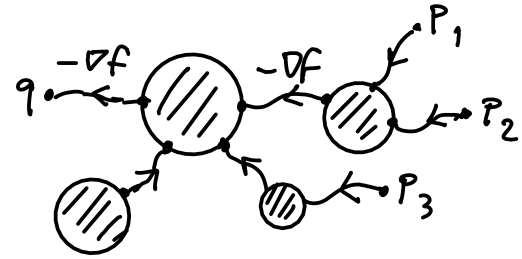
  (exception for m_1: don't count constant , instead ∂C as a chain)

- more generally, if L_0, L_1 have "clean intersection", ie $L_0 \cap L_1$ smooth and L_0, L_1 transverse in normal direction to $L_0 \cap L_1$, want to set $CF^*(L_0, L_1) = C_*(L_0 \cap L_1; \Lambda)$ & use chain as incidence condition at strip-like end — analytical details not completely clear.

c) <u>Cornea-Lalonde</u> approach to $CF^*(L,L)$ ("<u>clusters</u>")

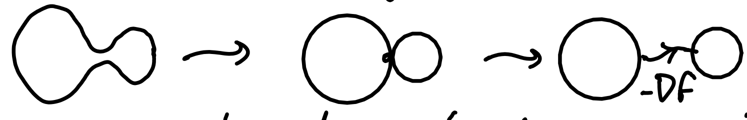
fix a Morse function $f: L \to \mathbb{R}$, then $CF^*(L,L) = \Lambda^{\operatorname{crit} f}$

m_k counts "clusters" of J-holom. discs + gradient flow lines



This cluster contributes to coeff^t of $T^{\text{area}} q$ in $m_3(p_1, p_2, p_3)$

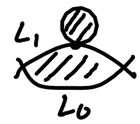
Now bubbling of discs is no longer a boundary of moduli space:



Instead, broken Morse trajectories are boundaries ($\rightsquigarrow A_\infty$-eqns (?)).

<u>Discs and obstruction</u>:

We've seen: if L_0 or L_1 bounds holom. discs then $\partial^2 \neq 0$ because index 2 moduli space has ends

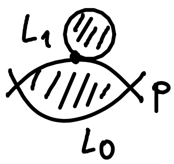


besides

Count contributions of such discs $\rightsquigarrow m_0 \in CF^*(L,L)$

In FOOO's theory: $m_0 = \sum_{\beta \neq 0} ev_* \left[\overline{\mathcal{M}}_{0,1}(X, L, J, \beta)\right] T^{\omega(\beta)}$

Then



is $m_2(m_0^{L_1}, p)$ and we get

$$m_1(m_1(p)) \pm \underbrace{m_2(m_0, p)}_{\text{of } L_1} \pm \underbrace{m_2(p, m_0)}_{\text{of } L_0} = 0$$

Hence, m_0 = obstruction to $\partial^2 = 0$

More generally, A_∞-equations hold if we include m_0 terms:

$\sum_{k,l \geq 0} \pm m_k(\dots, m_l(\dots), \dots) = 0$. This is called a "<u>curved A_∞-category</u>".

(& pretty hard to work with...)

Say L is unobstructed if $m_0 = 0$, weakly unobstructed if m_0 = mult. of 1.

($\Rightarrow$ central, so $m_1^2 = 0$ on $CF(L,L)$)

weakly unobstructed Lagrangians of a given "charge" form an honest A_∞-category

5

→ FOOO: try to cancel obstruction by deforming by $b \in CF^1(L,L)$:

on $CF^*(L,L)$, $m_k^b(c_k \dots c_1) = \sum m_{k+\ell}(b\dots b, c_k, b\dots b, c_{k-1}, \dots, c_1, b\dots b)$

still a curved A_∞-algebra; look for b s.t. $m_0^b = m_0 + m_1(b) + m_2(b,b) + \dots = 0$

or mult. of 1 so $(m_1^b)^2 = 0$: such b = "(weak) bounding cochain"

Then set objects = Lagrangians + equivalence classes of weak ∂ cochains