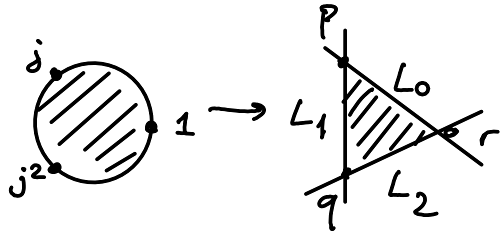
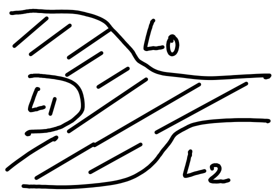
① Product structure : $CF^*(L_0,L_1)\otimes CF^*(L_1,L_2)\to CF^*(L_0,L_2)$

Look at $u: D^2\to M$ J-holom. disk with



$u(j)=p\in L_0\cap L_1$, $u(j^2)=q\in L_1\cap L_2$, $u(1)=r\in L_0\cap L_2$

$u([1,j])\subset L_0$, $u([j,j^2])\subset L_1$, $u([j^2,1])\subset L_2$

(or equivalently, u: 

$\to M$ Riem. surface of genus 0 with 3 strip-like ends [of finite energy])

Let $\mathcal{M}(p,q,r,[u],J)=\{$ such maps $\}$

expected dim. $=\mathrm{ind}([u])=\deg r-(\deg p+\deg q)$

(where trivialize u^*TM & pick graded Lits to define the degrees)

Then set $\quad q\cdot p=\sum_{\substack{r\in L_0\cap L_2\\ \phi\in\pi_2/\mathrm{ind}(\phi)=0}}\left(\#\mathcal{M}(p,q,r,\phi,J)\right)T^{\omega(\phi)}\,r$

Notes: → as usual, this is subject to achieving transversality, orientability ...

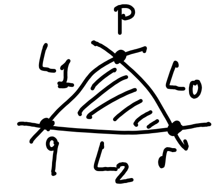
→ $\mathrm{Aut}(D^2)$ acts transitively on cyclically ordered triples of boundary points, so choice of $(1,j,j^2)$ is arbitrary.

→ lack of symmetry in deg p, q, r of index formula is because the degree of $r\in CF(L_0,L_2)$ is n minus that of $r\in CF(L_2,L_0)$

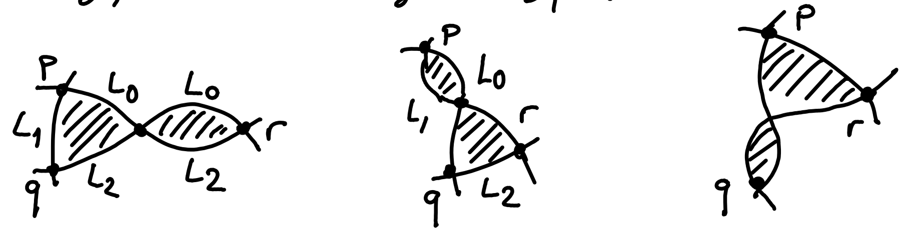
In general we have a "Poincaré duality" $CF^*(L,L')\simeq CF^{n-*}(L',L)^\vee$, compatible with differential, product, ...

Prop: If $[\omega].\pi_2(M,L_i)=0$ then the product satisfies Leibniz rule wrt differential, and hence induces a product on HF^*.

Moreover, the product on HF^* is associative.

Idea pf: (1) for Leibniz rule: consider index 1 moduli spaces 

compactify by adding limit configurations: in the absence of bubbling, those are of 3 types:



$\partial(q\cdot p)$ $\qquad$ $q\cdot(\partial p)$ $\qquad$ $(\partial q)\cdot p$

gluing theorem: assuming transversality, adding these gives a 1-manifold with boundary.

$\#$ends $= 0$ (w/ orientations, or mod 2) $\Rightarrow$ Leibniz identity.
(w/ signs depending on degree)

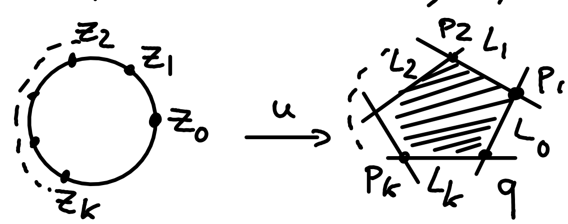
Thus:
- p, q closed $\Rightarrow \partial(q\cdot p) = \pm(\partial q)\cdot p \pm q\cdot(\partial p) = 0$
- ∂p exact, q closed $\Rightarrow q\cdot\partial p = \pm\,\partial(q\cdot p) \pm \underbrace{(\partial q)}_{0}\cdot p$ exact.

$\rightsquigarrow$ get product on HF^*

(2) associativity: we'll see now.

<u>Higher operations:</u> $CF^*(L_0,L_1)\otimes\cdots\otimes CF^*(L_{k-1},L_k) \xrightarrow{m_k} CF^*(L_0,L_k)[2-k]$
grading shift.

Look at J-hol. maps



D^2 with $(k+1)$ boundary marked pts
(Riem. surface w/ boundary, with $(k+1)$ strip-like ends)

exp. dim $\mathcal{M}(p_1\ldots p_k, q, [u], J) = \deg q - (\deg p_1 + \ldots + \deg p_k) + k - 2$

The term $k-2$ comes from the dim. of the moduli space of discs with $k+1$ marked points. Assume we can achieve transversality:

Then $m_k(p_k\ldots p_1) := \sum_{\substack{q\in L_0\cap L_k\\ [u]/\,\mathrm{ind}=0}} (\#\mathcal{M}(p_1\ldots p_k, q, [u], J))\, T^{\omega(\phi)}\, q$

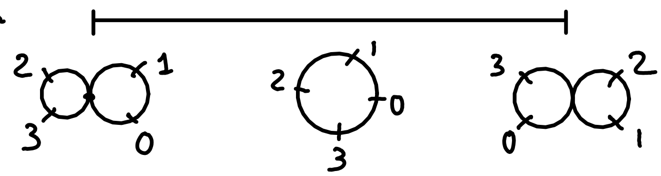
(m_1 = differential, m_2 = product).

③ NB: moduli space of discs with $(k+1)$ boundary marked points:

$$\mathcal{M}_{0,k+1} = \{(z_0,\dots,z_k) \in \partial D^2 \text{ distinct, in order}\} \quad \text{contractible, dim. } k-2$$

compactifies to moduli space $\overline{\mathcal{M}}_{0,k+1}$ of stable genus 0 Riem. surf. w/ one ∂ component & $k+1$ boundary marked pts, ie. trees of discs attached together at marked nodal points, s.t. each component has ≥ 3 special points

E.g: $\overline{\mathcal{M}}_{0,4}$ = closed interval



$\Rightarrow$ when considering sequences of holom. discs as above, limit configurations allowed by Gromov compactness =

- bubbling of spheres, of discs
- breaking of strips at marked pts

} (energy accumulates at various places in domain)

- degeneration of domain to $\partial\overline{\mathcal{M}}_{0,k+1}$

get relations when consider ∂ of 1-dim! families of discs.

Prop: Assuming no bubbling of discs/spheres, we have $\forall m \geq 1$, $\forall p_i \in L_{i-1} \cap L_i$,

$$\sum_{\substack{k,\ell \geq 1 \\ k+\ell = m+1 \\ 0 \leq j \leq \ell-1}} (-1)^* m_\ell(p_m, \dots, p_{j+k+1}, m_k(p_{j+k}, \dots, p_{j+1}), p_j, \dots, p_1) = 0$$

where $* = \deg(p_1) + \dots + \deg(p_j) + j$

Ex: $m_1(m_1(p)) = 0$; $m_1(m_2(p,q)) + m_2(p, m_1(q)) + (-1)^{\deg q + 1} m_2(m_1(p), q)$

differential $\qquad$ Leibniz rule

next one: $m_1(m_3(p,q,r)) \pm m_2(m_2(p,q), r) \pm m_2(p, m_2(q,r))$

$\pm\, m_3(m_1(p), q, r) \pm m_3(p, m_1(q), r) \pm m_3(p, q, m_1(r)) = 0$

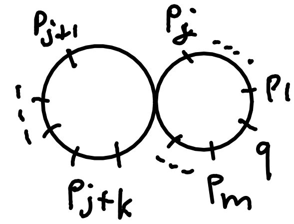
says: the product m_2 is associative up to homotopy (the homotopy being given by m_3).

& hence associative on cohomology.

... and so on.

④

Idea pf: consider a 1-dim^l moduli space $\mathcal{M}(p_1 \dots p_m, q; [u], J)$ and its ends:
Assuming transversality & absence of bubbling, limiting configs. are all of the form



(these are the codim-1 strata; configs with more components have higher codimension).

Total # ends $= 0 =$ sum of terms in the proposition
(coeff^t of $T^{\omega([u])} q$ in $\Sigma \dots$) $\square$

Def: A_∞-category = linear "category" (except associativity...) where morphism spaces are equipped with such algebraic operations $(m_k)_{k \geq 1}$

Fukaya category = A_∞-cat. with objects = Lagrangians
morphisms = Floer complexes
alg operations = as above.

(many versions with different details – we'll see later).

So far we have at best an A_∞-precategory i.e. morphisms and compositions are defined only for transverse objects.
$(CF(L,L) = ??)$

* At the homology level, the Donaldson-Fukaya category (hom = HF) is easier to work with but contains less information in general!