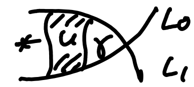
(1) <u>Lagrangian Floer homology:</u>

(M,ω) symplectic manifold $\supset L_0, L_1$ compact Lagrangian submanifolds

<u>Formally</u>, Floer homology = Morse theory for "action functional" on path space $\mathcal{P}(L_0,L_1)$, whose crit pts are constant paths & gradient flowlines = J-hol. strips.

More precisely: $\mathcal{A}: \underbrace{\widetilde{\mathcal{P}}(L_0,L_1)}_{\text{univ. cover}} \to \mathbb{R}$, $(\gamma,[u])$  $\mapsto \int u^*\omega$

$u:[0,1]^2\to M$ homotopy $*\to\gamma$

$$d\mathcal{A}(\gamma)\cdot v = \int_{[0,1]}\omega(\dot\gamma,v)\,dt = \int_{[0,1]} g(J\dot\gamma,v)\,dt = \langle J\dot\gamma, v\rangle_{L^2}$$

v: vect. field along γ, $v(0)\in T_{\gamma(0)}L_0$, $v(1)\in T_{\gamma(1)}L_1$

hence crit pts = const. paths $\dot\gamma=0$; gradient trajs. = J-hol. maps $\frac{\partial\gamma}{\partial s} = -J\dot\gamma$

Difficult to define rigorously ∞-dim! Morse theory, so use holom. curves instead.

<u>Actual setup:</u> Assume $L_0 \pitchfork L_1 \Rightarrow L_0\cap L_1$ finite set.

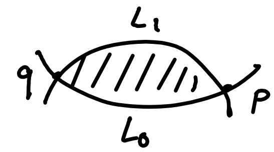
Recall Novikov ring $\Lambda = \{\sum a_i T^{\lambda_i} \,/\, \lambda_i\to+\infty\}$

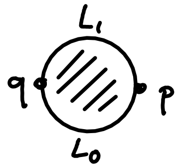
Floer complex $CF(L_0,L_1) = \Lambda^{|L_0\cap L_1|}$ free Λ-module gen^d by $L_0\cap L_1$.

<u>Goal:</u> define a differential ∂ by counting holomorphic discs:

<u>Look at:</u> $u: \mathbb{R}\times[0,1]\longrightarrow M$ equipped with J ω-compat. a.c.s.

s.t. $(*)$
- $\bar\partial_J u = 0$, ie. $\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$
- $u(s,0)\in L_0$, $u(s,1)\in L_1$
- $\lim_{s\to+\infty} u(s,t) = p$, $\lim_{s\to-\infty} u(s,t) = q$
- the energy $E(u) = \int u^*\omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 < \infty$



(NB: $\mathbb{R}\times[0,1] \underset{\text{bihol.}}{\simeq} D^2\setminus\{\pm1\}$ so also think of maps )

$\mathcal{M}(p,q,[u],J) = \{u \text{ sol}^{\text{ns}} \text{ of } (*)\}$ moduli space of holom. discs.

$[u]$: $\pi_2(M;L_0,L_1)$ homotopy class

$(*)$ is a Fredholm problem, exp. dim. $\mathcal{M} = \operatorname{ind}(\bar\partial_J)$

$\operatorname{ind}(\bar\partial_J) =$ <u>Maslov index</u>

comes from $\pi_1(\Lambda Gr) = \mathbb{Z}$ for Lagrangian grassmannian in $\mathbb{R}^{2n}$.

② Maslov index:

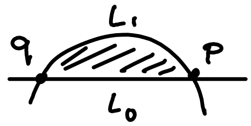
Let L_0, $L_1(t)_{t\in[0,1]}$ Lagr. subspaces of $\mathbb{R}^{2n}$, s.t. $L_1(0), L_1(1) \pitchfork L_0$

Then Maslov index of the path $L_1(t)$:= # times that $L_1(t)$ fails to be transverse to L_0 (counted with signs & multiplicities)

Ex. path $(e^{i\theta_1}\mathbb{R})\times\ldots\times(e^{i\theta_n}\mathbb{R})$ if $\theta_i \nearrow$ through 0, then $\mu(L_0, L_1(t)) = n$.

Now: given a strip u, trivialize $u^*TM \rightsquigarrow u^*TL_0, TL_1$ paths of Lagrangians. Can trivialize so that TL_0 remains constant

Then $\mathrm{ind}(u)$:= Maslov index of path TL_1 relative to TL_0 as one goes from p to q

Ex: 

in $\mathbb{R}^2$ has $\mathrm{ind}(u) = 1$.

- Want to define:

$$\partial(p) = \sum_{\substack{q\in L_0\cap L_1 \\ \phi\in\pi_2/\,\mathrm{ind}(\phi)=1}} \left(\#\mathcal{M}(p,q,\phi,J)/_{\mathbb{R}}\right) T^{\omega(\phi)}\, q$$

symplectic area of ϕ $(=\int u^*\omega)$

translation

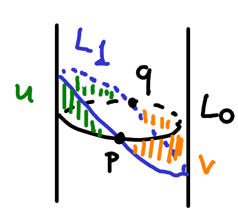
Issues: - transversality
- compactness, bubbling
- orientation of $\mathcal{M}$ (⇒ signed counts? else work over $\mathbb{Z}/2$)
- $\partial^2 = 0$?

A special case where bubbling isn't an issue:

Thm (Floer): if $[\omega].\pi_2(M) = 0$ and $[\omega].\pi_2(M, L_i) = 0$ then ∂ is well-def^d, $\partial^2 = 0$, and HF is indep^t of chosen J & invariant under Hamiltonian deformations of L_0 and/or L_1

Corollary: $[\omega].\pi_2(M,L) = 0$, ψ Ham. diffeo, $\psi(L) \pitchfork L \Rightarrow |\psi(L)\cap L| \geq \sum b_i(L)$

(special case of Arnold's conjecture; idea: $HF(L,\psi(L)) \simeq H^*(L)$; rank CF ≥ rank HF)

Example: $T^*S^1 = \mathbb{R}\times S^1$ 

$CF(L_0, L_1) = \Lambda p \oplus \Lambda q$

$\partial p = (T^{\mathrm{area}(u)} - T^{\mathrm{area}(v)})\, q$

$\partial q = 0$.

③ In this case $\exists$ well-def^{d} $\mathbb{Z}$-grading (Maslov index only depends on p, q, not on homotopy class of strip), e.g. $\deg(p) = 0$, $\deg(q) = 1$.

2 cases:
* $\mathrm{area}(u) = \mathrm{area}(v)$ (L_0, L_1 Ham.-isotopic): $HF(L_0, L_1) \simeq H^*(S^1; \Lambda)$
* $\mathrm{area}(u) \neq \mathrm{area}(v)$ (L_0, L_1 can be disjoined): $HF(L_0, L_1) = 0$.

<u>Back to issues with the definition of ∂:</u>

- transversality is achieved for simple maps by picking generic J.
  For multiply covered maps (or configs with complicated bubbling), need various tricks... (domain-dependent J's, multivalued perturbations, virtual cycles, ...)
- orientation on moduli space: need auxiliary data, & top. assumption on L_i.
  Namely: if equip L_i with <u>spin</u> structures (ie. double cover of frame bundle) then we get an orientation on moduli spaces.