Math 253y – Homework 3 – due Thursday November 29, 2018.

Turn in: as much as you are able to complete within a reasonable amount of time.

Homework policy: if you are taking this course for a grade, you are expected to submit your own work. You are welcome to collaborate with other students, and you are encouraged to consult with the instructor or with the CA, or to look up references as needed, but you are expected to write up your own arguments and cite your sources. It is ok for your answer to follow the outline of an argument found in a textbook or on Math Overflow if you attribute the original source; it is not ok to copy someone else's proof verbatim without attribution.

1. The goal of this problem is to prove Gromov's nonsqueezing theorem: if the closed ball $B(r)$ of radius r in $(\mathbb{R}^{2n},\omega_0)$ admits a symplectic embedding into the cylinder $(D^2\times\mathbb{R}^{2n-2},\omega_0)$ (where $D^2\subset\mathbb{R}^2$ is the closed unit disc), then $r\le 1$.

a) (Gromov's monotonicity lemma) Let C be a holomorphic curve in $\mathbb{R}^{2n}$ equipped with its standard complex structure J_0, and whose boundary lies outside of the ball $B(r_0)$. Assume that the origin lies on C and is a smooth point of C. For $r\in(0,r_0)$, denote by $A(r)$ the symplectic area of $C\cap B(r)$; and when C intersects the sphere $S(r)=\partial B(r)$ transversely, denote by $L(r)$ the length of the curve $\gamma(r)=C\cap S(r)$. By considering the 1-form $\lambda=\frac{1}{2}\sum x_i dy_i - y_i dx_i$ and its integral over $\gamma(r)$, show that

$$\frac{dA}{dr}\ge L(r)\ge \frac{2A(r)}{r}$$

for almost every r, and deduce that $A(r)\ge \pi r^2$.

b) Consider $M=(S^2)^n$, equipped with a product symplectic form ω_N where the first factor has area π and the other factors have area $N\pi$, where $N\ge 1$ is an integer. By first considering the case of the standard product complex structure, show that for any ω_N-compatible almost complex structure J and any point $p\in M$, there exists a J-holomorphic sphere passing through p and representing the homology class of the first factor sphere, $A=[S^2\times pt]\in H_2(M,\mathbb{Z})$. (Where did you use the assumption that N is integer?)

c) Assume $(B(r),\omega_0)$ admits a symplectic embedding into $(D^2\times\mathbb{R}^{2n-2},\omega_0)$. First show that for $\varepsilon>0$ arbitrarily small and N sufficiently large, there is a symplectic embedding $\varphi:(B(r-\varepsilon),\omega_0)\hookrightarrow(M,\omega_N)$. Equip M with an ω_N-compatible almost-complex structure which agrees with $\varphi_*(J_0)$ over $\varphi(B(r'))$ (where $r'=r-2\varepsilon$), and use the result of part (b) for $p=\varphi(0)$ together with the monotonicity lemma of part (a) to show that $r'<1$. (Why can we remove the smoothness assumption from the monotonicity lemma?) Deduce that $r\le 1$.

2. Let $M=\{(x,y,z)\in\mathbb{C}^3\,|\,xy=P(z)\}$, where $P(z)=\prod_{i=1}^k(z-\lambda_i)$ is a degree k polynomial with distinct roots, equipped with the restriction of the standard symplectic form of $\mathbb{C}^3$. Fix $r\ne|\lambda_i|$, and let $L=\{(x,y,z)\in M\,|\,|x|=|y| \text{ and } |z|=r\}$.

a) Show that L is a Lagrangian torus in M, and that the Maslov index of a disc with boundary on L is equal to twice its intersection number with $D=\{z=0\}\subset M$. (Hint: one way to proceed is to show that $\Omega = (xz)^{-1}dx \wedge dz = -(yz)^{-1}dy \wedge dz$ is a well-defined holomorphic 2-form on $M \setminus D$ and induces a complex trivialization of $\wedge^2 TM$ which is compatible with the real subbundle $\wedge^2 TL$ over L.)

b) Let f be a meromorphic function on the unit disc, such that $|f| = 1$ everywhere on the unit circle, with zeros and poles at a finite subset $\{z_i,\ 1 \le i \le \ell\} \subset D^2$ and with order $\alpha_i \in \mathbb{Z}$ at each z_i. Show that there exists a constant θ such that

$$f(z) = e^{i\theta} \prod_{i=1}^{\ell} \left(\frac{z - z_i}{1 - \bar{z}_i z} \right)^{\alpha_i}.$$

c) Returning to the setup of part (a), show that L does not bound any holomorphic discs of Maslov index zero, and that there are 2^ℓ holomorphic discs of Maslov index 2 with boundary on L passing through any given point of L, where $\ell = \#\{i \,|\, |\lambda_i| < r\} \in \{0, \dots, k\}$. (Hint: consider the quantities xy and x/y). Also show that these discs are regular.

d) Show that L is monotone, and that in a suitable basis of $H_1(L, \mathbb{Z}) \simeq \mathbb{Z}^2$ the disc-counting superpotential

$$W_L = \sum_{\substack{\beta \in H_2(M,L) \\ \mu(\beta)=2}} n_\beta \, z^{\partial\beta} \in \mathbb{Z}[H_1(L)], \tag{1}$$

where n_β is the degree of the evaluation map $ev : \mathcal{M}_{0,1}(L, \beta) \to L$ for holomorphic discs with one boundary marked point, is given by

$$W_L = z_1(1 + z_2)^\ell.$$

Note: this shows that M contains at least $k + 1$ different types of monotone Lagrangian tori. In the case $k = 1$, M is symplectomorphic to the standard $\mathbb{R}^4$, and these tori are product and Chekanov tori. In the case $k = 2$, M is symplectomorphic to T^*S^2, and the tori corresponding to $\ell = 2$ are known as Albers-Frauenfelder tori.

3. Let L be a monotone Lagrangian torus in a monotone symplectic manifold (M^{2n}, ω) (compact or convex at infinity) (i.e. the symplectic area of discs in (M, L) is positively proportional to their Maslov index). The goal of this problem is to give a criterion for the (non)vanishing of the Floer cohomology of L (twisted by a local coefficient system).

Fix a compatible almost-complex structure J, and assume that for every class $\beta \in \pi_2(M, L)$ with $0 < \mu(\beta) \le n + 1$ the (compactified) moduli space $\overline{\mathcal{M}}_{0,k+1}(L, \beta)$ of J-holomorphic discs representing the class β with $k + 1$ boundary marked points (labelled $0, \dots, k$) is regular, and that the evaluation map $ev_{0,\beta} : \overline{\mathcal{M}}_{0,k+1}(L, \beta) \to L$ at the 0-th marked point is a submersion with compact (oriented) fibers (with boundary and corners).

In the *de Rham model*, the Lagrangian Floer cochain complex of L twisted by a rank 1 local coefficient system $\xi \in \hom(\pi_1(L), \mathbb{C}^*) \simeq H^1(L, \mathbb{C}^*)$ is given by

$$CF^*(L, \xi) := \Omega^*(L, \mathbb{C}) \otimes \mathbb{C}[t^{\pm 1}]$$

(Laurent polynomials in a formal variable t with $\deg(t) = 2$, whose coefficients are complex-valued differential forms on L), and the Floer differential is defined by

$$\partial\alpha = d\alpha + \sum_{\beta\neq 0} \xi(\partial\beta)\, t^{\mu(\beta)/2}\, (ev_{0,\beta})_*(ev_{1,\beta}^*\alpha),$$

involving evaluation maps from (compactified) moduli spaces of discs with two boundary marked points, and where the pushforward is defined by integration along the fibers.[1] (The Floer products $\mathfrak{m}_k$ can be defined similarly using compactified moduli spaces of discs with $k+1$ marked points, though this requires regularity over a greater range of Maslov indices: $\mathfrak{m}_k(\alpha_k, \dots, \alpha_1) = \sum_\beta \xi(\partial\beta) t^{\mu(\beta)/2} (ev_{0,\beta})_* \left(ev_{1,\beta}^*\alpha_1 \wedge \dots \wedge ev_{k,\beta}^*\alpha_k\right)$, where the sum is over all β, except when $k = 0$ where we disallow $\beta = 0$.)

We will admit without proof that $\partial^2 = 0$ (optional: check this!); more generally, the operations $\mathfrak{m}_k$ satisfy the A_∞-relations.[2]

Meanwhile, the disc-counting superpotential $W_L \in \mathbb{Z}[H_1(L)]$ defined by (1) above can be viewed as a complex-valued function of $\xi \in H^1(L, \mathbb{C}^*)$, namely

$$W_L(\xi) = \sum_{\mu(\beta)=2} n_\beta\, \xi(\partial\beta) \in \mathbb{C},$$

where $n_\beta = \deg(ev_{0,\beta} : \overline{\mathcal{M}}_{0,1}(L, \beta) \to L) \in \mathbb{Z}$. (Note that $\mathfrak{m}_0 = tW_L(\xi) \in CF^2(L, \xi)$). We define the *logarithmic derivative*

$$\nabla W_L = \frac{\partial W_L}{\partial \log \xi} = \sum_{\mu(\beta)=2} n_\beta\, \xi(\partial\beta)\, [\partial\beta] \in H_1(L, \mathbb{C}).$$

(Optional: check that this is indeed the differential of W_L as a complex-valued function of $\log\xi \in H^1(L, \mathbb{C})$).

(a) Let $\alpha \in \Omega^k(L, \mathbb{C})$ be a closed k-form. Show that $\partial\alpha = t\beta + O(t^2)$, where β is a closed $(k-1)$-form, and $[\beta] \in H^{k-1}(L, \mathbb{C})$ is obtained by interior product of $[\alpha] \in H^k(L, \mathbb{C})$ with $\nabla W_L \in H_1(L, \mathbb{C})$. (Namely: let $V = H_1(L, \mathbb{C})$, since $L \simeq T^n$, using cup-product we have a canonical isomorphism $H^k(L, \mathbb{C}) \simeq \wedge^k H^1(L, \mathbb{C}) = \wedge^k V^*$, and the interior product $V \otimes \wedge^k V^* \to \wedge^{k-1} V^*$.) (Hint: for $\mu(\beta) = 2$, what are the fibers of $ev_{0,\beta}$?)

[1] Given a submersion $p : X \to Y$ with compact fibers of dimension d, a k-form α on X, given a point $y \in Y$ and vectors $v_1, \dots, v_{k-d} \in T_yY$, by definition

$$(p_*\alpha)_y(v_1, \dots, v_{k-d}) = \int_{p^{-1}(y)} \iota_{v_{k-d}^\#} \cdots \iota_{v_1^\#}\alpha,$$

where $v_i^\#$ is any vector field along $p^{-1}(y)$ such that $dp(v_i^\#) = v_i$ everywhere. Stokes' theorem implies that $d(f_*\alpha) = f_*(d\alpha) + (-1)^{\deg f_*\alpha}(f_{|\partial X})_*(\alpha_{|\partial X})$.

[2] This is basically a verification involving Stokes' theorem, properties of pushforward by the evaluation map, and the structure of the boundary strata of $\overline{\mathcal{M}}_{0,k+1}(L, \beta)$. See e.g. Proposition 2.5 in Solomon-Tukachinsky's preprint arXiv:1608.01304 (case $l = 0$ since we have no interior marked points).

(b) Show that if $\nabla W_L \neq 0$ then every ∂-closed element of $CF^*(L,\xi)$ is ∂-exact, i.e. $HF^*(L,\xi) = H^*(CF^*(L,\xi),\partial) = 0$. Hint: first show that the leading order term (with the lowest power of t) of an element of $\mathrm{Ker}(\partial)$ is closed, and that its cohomology class lies in the image of interior product with ∇W_L.

(c) Show that, when $\dim M = 2n \leq 4$, if $\nabla W_L = 0$ then $HF^*(L,\xi) \simeq H^*(L,\mathbb{C}) \otimes \mathbb{C}[t^{\pm 1}]$. (This in fact remains true in higher dimensions, using more sophisticated arguments.)

Corollary: if the Laurent polynomial W_L (viewed as a function on $H^1(L,\mathbb{C}^*) \simeq (\mathbb{C}^*)^n$) has a critical point, then L cannot be displaced from itself by a Hamiltonian isotopy (by invariance of Floer homology).

Optional: use this to show that the Albers-Frauenfelder monotone torus in T^*S^2 (see Problem 2) is non-displaceable. (The same holds for all the tori of Problem 2 with $k \geq \ell \geq 2$. What about the remaining cases $\ell = 0$ and $\ell = 1$?)

4. Let L be the equator in S^2 equipped with its standard symplectic form. Calculate the Floer cohomology $HF^*(L,L)$ and its product structure, using any reasonable approach. Feel free to make any reasonable assumptions about orientations of moduli spaces. Is the Floer cohomology isomorphic as a ring to the classical cohomology?

Possible approaches include, but are not limited to:

- using invariance under Hamiltonian isotopies (since L is monotone), we can consider three different copies L_0, L_1, L_2 of the equator, pairwise intersecting transversely in two points. Look for holomorphic bigons and triangles with corners at given intersection points to calculate the Floer differential on $CF^*(L_i, L_j)$ and the Floer product on $CF^*(L_1,L_2) \otimes CF^*(L_0,L_1)$. [3]
- use the de Rham model of $HF^*(L)$ described in Problem 3. (with the trivial local system $\xi = 1$).
- learn about the "pearly trees" model of Lagrangian Floer cohomology, for example see Biran and Cornea's paper "Lagrangian quantum homology" (arXiv:0808.3989), and apply it to a Morse function on S^1 with two critical points.

Optional: try more than one way and compare the calculations.

[3] Consistent signs can be determined using a rule due to Seidel [Section 13 of "Fukaya categories and Picard-Lefschetz theory"]: fix orientations of the Lagrangians L_i. Then the sign of a polygon can be taken to be $(-1)^r$, where r is the number of corners at which the corresponding Floer generator in $CF^*(L_i, L_j)$ has odd degree *and* the positive orientation of the boundary of the polygon disagrees with the chosen orientation of L_j.