Math 253y – Homework 2 – due Thursday October 18, 2018.

REVISED 10/9: removed last part of 1.b (the given formulas for horizontal lifts of radial and angular vectors were incorrectly scaled outside of S_a and weren't needed anyway).

Turn in: all except at most 3 parts of one problem.

Homework policy: if you are taking this course for a grade, you are expected to submit your own work. You are welcome to collaborate with other students, and you are encouraged to consult with the instructor or with the CA, or to look up references as needed, but you are expected to write up your own arguments and cite your sources. It is ok for your answer to follow the outline of an argument found in a textbook or on Math Overflow if you attribute the original source; it is not ok to copy someone else's proof verbatim without attribution.

1. Consider the $(n-1)$-dimensional affine quadric $Q_a = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum z_i^2 = a\}$ ($a \in \mathbb{C}^*$) equipped with the restriction of the standard symplectic form of $\mathbb{C}^n$.

a) We identify T^*S^{n-1} with $\{(p,q) \in \mathbb{R}^n \times \mathbb{R}^n,\ |q| = 1 \text{ and } \langle p, q\rangle = 0\}$, with the standard symplectic form $\omega_0 = \sum dp_i \wedge dq_i$. Show that, for $a \in \mathbb{R}_+$, the map

$$z \mapsto (-\mathrm{Im}(z)\,|\mathrm{Re}(z)|, \mathrm{Re}(z)/|\mathrm{Re}(z)|)$$

defines a symplectomorphism from Q_a to T^*S^{n-1}, under which the the zero section of T^*S^{n-1} is identified with the Lagrangian sphere $S_a = Q_a \cap \mathbb{R}^n$.

b) Consider the symplectic Lefschetz fibration (with an isolated singularity at the origin) $\pi : \mathbb{C}^n \to \mathbb{C}$ given by $\pi(z_1, \dots, z_n) = \sum z_i^2$, with fibers $\pi^{-1}(a) = Q_a$. Recall that the horizontal distribution $H_z \subset T_z\mathbb{C}^n$ at $z \in Q_a$ is the symplectic orthogonal to TQ_a, and that the horizontal lift of a vector $v \in T_a\mathbb{C}$ is the unique vector $v^\# \in H_z$ such that $d\pi(v^\#) = v$.

Show that the horizontal distribution H_z is the complex line spanned by the complex conjugate $\bar{z}$.

c) Show that horizontal parallel transport (i.e. following the horizontal lifts of the tangent vectors to an arc in $\mathbb{C}$) maps the Lagrangian spheres $S_a = Q_a \cap (e^{i\theta/2}\mathbb{R})^n \subset Q_a$ to each other. (Here $\theta = \arg(a)$). Also show that the image of S_a under horizontal parallel transport over the segment $[0, a] \subset \mathbb{C}$ extends over the origin and gives a smooth Lagrangian n-disc $D_a \subset \mathbb{C}^n$ with boundary S_a.

d) Use parallel transport to give an example of a Lagrangian $S^1 \times S^{n-1}$ in $\mathbb{C}^n$; as well as, for odd n, a non-orientable Lagrangian S^{n-1}-bundle over S^1.

e) Now we consider the n-dimensional quadric $\hat{Q} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum z_i^2 = 1\}$ (symplectomorphic to T^*S^n by part (a)), and the Lefschetz fibration $\hat{\pi} : \hat{Q} \to \mathbb{C}$ given by $\pi(z_1, \dots, z_{n+1}) = z_{n+1}$, with fibers $\hat{\pi}^{-1}(a) \cong Q_{1-a^2}$, and two critical points $(0, \dots, 0, \pm 1)$. Show that the horizontal distribution $\hat{H}_{\hat{z}}$ at $\hat{z} = (z_1, \dots, z_{n+1}) \in \hat{Q}$ maps under the projection to the first n coordinates to $H_{(z_1,\dots,z_n)} = \mathrm{span}_{\mathbb{C}}(\bar{z}_1, \dots, \bar{z}_n)$. Conclude that parallel transport between the fibers of $\hat{\pi}$ maps the Lagrangian spheres $S_{1-a^2} \subset Q_{1-a^2} \simeq \hat{\pi}^{-1}(a)$ to each other.

f) What is the Lagrangian submanifold of $\hat{Q} \simeq T^*S^n$ traced out by parallel transport of these Lagrangian spheres over the interval $[-1, 1]$? over $[1, +\infty)$?

2. Consider $S^2 \times S^2$ (viewed as the space of pairs of unit vectors $v_1, v_2 \in \mathbb{R}^3$) with the standard product symplectic form.

a) Denoting the components of $v_i = (x_i, y_i, z_i)$, show that the functions $f(v_1, v_2) = z_1 + z_2$ and $g(v_1, v_2) = \|v_1 + v_2\|^2 = 2 + v_1 \cdot v_2$ define a completely integrable system on $S^2 \times S^2$. Describe geometrically the Hamiltonian vector fields X_f and X_g.

b) Show that f and $\sqrt{g}$ generate S^1-actions (away from $g = 0$); describe the T^2-orbits geometrically.

c) What is the base of the integrable system $(f, \sqrt{g})$? Describe the fibers at the vertices.

3. The Thurston manifold M is the quotient of $(\mathbb{R}^4, \omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$ by identifying $(x_1, x_2, x_3, x_4) \sim (x_1+1, x_2, x_3, x_4) \sim (x_1, x_2+1, x_3+x_4, x_4) \sim (x_1, x_2, x_3+1, x_4) \sim (x_1, x_2, x_3, x_4 + 1)$.

a) Verify that M is a compact symplectic manifold, and projection to (x_1, x_2) realizes M as the total space of a symplectic fibration with T^2-fibers over T^2.

b) Show that $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}^3$. (In particular M cannot admit a compatible integrable complex structure, since compact Kähler manifolds have even-dimensional H_1).

Hint: this can be done either directly, or by packaging (x_2, x_3, x_4) into the off-diagonal entries of an upper-triangular 3×3 matrix with 1's on the diagonal, and thinking of the group of such matrices with integer coefficients.

Note: M is diffeomorphic to a compact complex surface, the *Kodaira surface*, so admits an integrable complex structure, which is however not compatible with any symplectic form.

c) Show that the projections to (x_1, x_4) and to (x_2, x_4) give two Lagrangian torus fibrations on M. For each of these, describe the affine structure on the base and determine whether there exists a Lagrangian section. (How do these fibrations differ from the trivial fibration with total space the 4-torus?)

4. Let (M, ω) be a symplectic manifold, J a compatible almost-complex structure, and g the corresponding Riemannian metric. Show that two-dimensional almost-complex submanifolds of M are absolutely volume minimizing in their homology class, i.e.: let C, C' be two-dimensional compact closed oriented submanifolds of M, representing the same homology class $[C] = [C'] \in H_2(M, \mathbb{Z})$. Assume that $J(TC) = TC$. Then $\mathrm{vol}_g(C) \leq \mathrm{vol}_g(C')$.