Math 253y – Homework 1 – due Thursday September 27, 2018.

Turn in any 4 out of the 5 problems below.

Homework policy: if you are taking this course for a grade, you are expected to submit your own work. You are welcome to collaborate with other students, and you are encouraged to consult with the instructor or with the CA, or to look up references as needed, but you are expected to write up your own arguments and cite your sources. It is ok for your answer to follow the outline of an argument found in a textbook or on Math Overflow if you attribute the original source; it is not ok to copy someone else's proof verbatim without attribution.

1. Let N be a coisotropic submanifold in a symplectic manifold (M, ω). (i.e., at every point $p \in N$, $(T_pN)^{\perp\omega} \subseteq T_pN$). It is a general fact that $(TN)^{\perp\omega}$ is closed under Lie bracket, and hence defines an integrable *foliation* on N, called the isotropic foliation of N.

Assume that N is a *fibered coisotropic* submanifold, i.e. there exists a locally trivial fibration $\pi : N \to Q$ with connected fibers, whose fibers are the isotropic leaves. Show that Q carries a natural symplectic form $\tilde{\omega}$ such that $\omega_{|N} = \pi^*\tilde{\omega}$.

Hint: first show that, if W is a coisotropic subspace in a symplectic vector space (V, Ω), then Ω induces a natural symplectic structure on the quotient $W/W^{\perp\Omega}$.

2. A *Lagrangian correspondence* between symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a Lagrangian submanifold of $M_1^- \times M_2 = (M_1 \times M_2, -\pi_1^*\omega_1 + \pi_2^*\omega_2)$. As a special case, a Lagrangian submanifold of M is a Lagrangian correspondence between the point and M.

The *geometric composition* of $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$ is defined to be

$$\begin{aligned} L_{12} \circ L_{01} &= \pi_{02}\big((L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)\big) \\ &= \{(x, z) \in M_0^- \times M_2 \,|\, \exists y \in M_1, (x, y, y, z) \in L_{01} \times L_{12}\}. \end{aligned} \tag{1}$$

a) Show that, if the composition is *transverse and embedded*, i.e.

1. $L_{01} \times L_{12}$ is transverse to the diagonal $M_0 \times \Delta_{M_1} \times M_2$ in $M_0 \times M_1 \times M_1 \times M_2$, and
2. the projection π_{02}, restricted to the intersection, is an embedding (in particular it is injective, i.e. y in equation (1) is at most unique for given x and z),

then $L_{12} \circ L_{01}$ is a Lagrangian correspondence in $M_0^- \times M_2$.

b) With the same notations as in Problem 1, assume $N \subset (M, \omega)$ is a fibered coisotropic submanifold over a base Q. Show that the graph of the projection $\pi : N \to Q$ defines a Lagrangian correspondence between Q and M, and that the composition of N with any Lagrangian submanifold of Q (viewed as a correspondence from $\{pt\}$ to Q) is always transverse and embedded.

Corollary: Recall that, if M carries a Hamiltonian G-action with moment map $\mu : M \to \mathfrak{g}^*$, with 0 a regular value, then $\mu^{-1}(0)$ is coisotropic, and fibered over the reduced space $M^{red} = \mu^{-1}(0)/G$. It then follows that the preimage of a Lagrangian submanifold of M^{red} in $\mu^{-1}(0)$ is a Lagrangian submanifold of M.

3. We consider the family of tori in $(\mathbb{C}^2, \omega_0)$ defined by

$$L_{r,\lambda} = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1 z_2 - 1| = r,\ |z_1|^2 - |z_2|^2 = \lambda\}$$

for $(r, \lambda) \neq (1, 0)$. For $r < 1$, $L_{r,0}$ is an instance of the monotone Chekanov torus.

a) Show that $L_{r,\lambda}$ is a Lagrangian torus.

b) Show that, for $\lambda \neq 0$, $L_{r,\lambda}$ is Hamiltonian isotopic to a product torus $S^1(r_1) \times S^1(r_2) = \{(z_1, z_2) \,|\, |z_1| = r_1,\ |z_2| = r_2\} \subset \mathbb{C}^2$ (for suitable r_1, r_2). What about the case $\lambda = 0$?

4. Consider the sphere S^n equipped with its standard metric (as the unit sphere in $\mathbb{R}^{n+1}$), fix $\epsilon > 0$, and let $f : \mathbb{R}_+ \to \mathbb{R}$ be a smooth concave function such that $f(t) = t$ for $0 < t < \frac{\epsilon}{2}$ and $f(t) = \epsilon$ (or any other constant) for $t > 2\epsilon$. Define a Hamiltonian on the complement of the zero section in the cotangent bundle, $H : T^*S^n \setminus S^n \to \mathbb{R}$, by $H(x, p) = f(|p|)$ for all $x \in S^n$ and $p \in T^*_x S^n \setminus \{0\}$. Let ϕ be the time π flow generated by the Hamiltonian vector field of H.

a) Show that ϕ extends to a symplectomorphism of T^*S^n which maps the zero section to itself, and agrees with identity for $|p| > 2\epsilon$.

b) Show that ϕ is in general not a Hamiltonian diffeomorphism of T^*S^n (e.g. consider n even).

c) Sketch a proof that, for $n = 2$, ϕ^2 is smoothly isotopic to identity as a compactly supported diffeomorphism (i.e. there is a path of diffeomorphisms which agree with identity away from a neighborhood of the zero section, connecting identity to ϕ^2).

Hint: given $x \in S^2$ and a unit covector $p \in T^*_x S^2$, describe the image under ϕ^2 of the arc $\gamma(t) = (x, tp)$, $t \in [0, 2\epsilon]$, then describe a choice of a homotopy (rel. end points) between the arcs $\phi^2(\gamma)$ and γ. Then check that this homotopy can be performed simultaneously for all (x, p) and gives the desired result.

(Note: I do not necessarily want to see explicit formulas for your homotopy, a picture sketch and a short description suffice; but you should convince yourself that your construction really works. What are the main things to check? Here again you don't need to write out the complete details of a proof.)

Context: Given a Lagrangian sphere $L \simeq S^n$ in a symplectic manifold (M, ω), the Weinstein neighborhood theorem gives an embedding of a neighborhood of the zero section $U \subset T^*S^n$ into M. Choosing ϵ small enough, we can arrange for ϕ in the above construction to be supported inside U, and extend it by identity to obtain a symplectomorphism of M, called the *Dehn twist* about L. When M is compact and $[L] \neq 0 \in H_n(M)$, the action of the Dehn twist τ_L on the homology of M has order 2 for even n, and is of infinite order for odd n. For $n = 2$ the square τ_L^2 is smoothly isotopic to identity as a diffeomorphism, but it is in general not isotopic to identity among symplectomorphisms (as first shown by Seidel).

5. Given a Lagrangian immersion $i : L \to (M, \omega)$ of a compact closed n-manifold into a symplectic $2n$-manifold (i.e. a smooth immersion such that $i^*\omega = 0$), and denoting by $j : S^1 \to \mathbb{R}^2$ the standard embedding of a small circle, it was first shown by Audin, Lalonde and Polterovich that the immersion $i \times j : L \times S^1 \to M \times \mathbb{R}^2$ can be deformed to a Lagrangian embedding. An explicit construction goes as follows.

Assume that $i(L)$ has at most double points (this can be ensured by a small perturbation), and let $f : L \to \mathbb{R}$ be a smooth function whose values at the two preimages of a double point are always different.

a) Show that the diffeomorphism of $T^*L \times \mathbb{R}^2$ given by

$$\sigma(x, \xi, s, t) = (x, \xi + s\, df(x), s, t - f(x))$$

preserves the standard symplectic form of $T^*L \times \mathbb{R}^2$.

b) An easy adaptation of Weinstein's Lagrangian neighborhood theorem gives an immersion ϕ from a neighborhood of the zero section in T^*L to M which agrees with i on the zero section and such that $\phi^*\omega$ agrees with the standard symplectic form of T^*L.

Prove that the map from $L \times S^1$ to $M \times \mathbb{R}^2$ given by $(x, \theta) \mapsto (\phi \times \mathrm{id}) \circ \sigma(x, 0, \epsilon \cos\theta, \epsilon \sin\theta)$ is well-defined and a Lagrangian embedding for sufficiently small ϵ.