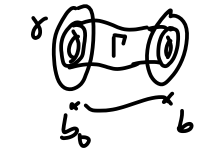
The geometry of Lagrangian torus fibrations

* Assume $\pi : (M^{2n}, \omega) \longrightarrow B^n$ (proper map) has Lagrangian fibers over regular values of π. $F_b = \pi^{-1}(b)$

 - Flux gives a local chart $U_{b_0} = \text{nbd of } b_0 \text{ in } B \xrightarrow{\phi} H^1(F_{b_0}, \mathbb{R})$.
 Abstractly: nbd of $F_{b_0} \simeq$ nbd of zero section in $T^*F_{b_0}$, and for b close to b_0, F_b = graph of a closed 1-form α_b. $\phi(b) = [\alpha_b] \in H^1(F_{b_0}, \mathbb{R})$.
 Explicitly: $\forall \gamma \in H_1(F_{b_0})$, $\langle \phi(b), \gamma \rangle$ = symplectic area of cylinder (indept of choices by Stokes)



 - This map is always a local embedding: injectivity: if $\phi(b) = \phi(b') \Rightarrow \alpha_b - \alpha_{b'} = df$, but $F_b \cap F_{b'} = \phi \Rightarrow f$ has no critical points, contradiction.
 similarly, differential at zero: $v \in T_{b_0}B \Rightarrow v^\#$ any lift, $d\phi(v) = [-\iota_{v^\#}\omega_{|F_{b_0}}] \in H^1(F_{b_0}, \mathbb{R})$ can't be exact by the same argument (the closed 1-form $\iota_{v^\#}\omega$ has no zeros since $v^\#$ nowhere tangent to F_b)

 - So $\dim H^1(F_{b_0}) \geq n$. and $T^*F_{b_0}$ trivial ($d\pi : T^*F_{b_0} \xrightarrow{\sim} T_{b_0}B \times F_{b_0}$).
 Case of interest: regular fibers are tori. Then ϕ are local charts $B \supset U_{b_0} \to \mathbb{R}^n$.

 - Then we have an integer affine structure on B (or on regular locus $B^{reg} \subset B$)
 ie. local charts into $\mathbb{R}^n$, with transition functions in affine group $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$
 (translations = change reference fiber, $GL_n(\mathbb{Z})$ = change basis of $H_1(F)$).
 Equivalently, affine str = data of a family of lattices $\Lambda = TB^{\mathbb{Z}} \subset TB$.

* Connection to completely integrable systems = n commuting Hamiltonians $f_1, \dots, f_n : M \to \mathbb{R}$
 with $\{f_i, f_j\} = 0$; $df_1 \dots df_n$ linearly independent over regular values of $f = (f_1 \dots f_n) : M \to \mathbb{R}^n$.
 $X_i = X_{f_i}$ satisfy $df_j(X_i) = \omega(X_i, X_j) = 0$ ie. the vf's X_i are tangent to fibers of f
 At a regular point, X_i linearly indept $\Rightarrow$ span tangent space to the fiber, which is hence Lagrangian.

 So: f is a Lagrangian fibration away from critical points.

 Ex. moment map $\mu : M \to \mathbb{R}^n$ on a toric sympl. mfld. In this case the affine structure on the base $= \mu(M) = \Delta \subset \mathbb{R}^n$ is exactly given by std coords. (up to 2π)

 Ex: $\pi : M \to B$ Lagr. fibration, $\phi : U \to \mathbb{R}^n$ ($U \subset B$) local coords. $\Rightarrow f = \phi \circ \pi$ locally integrable system!

Could do this for any loc. coords on ϕ, but advantageous to use affine coords / flux chart.

Q: a priori the flows of $X_i = X_{f_i}$ in a locally integrable system have no requirement to be periodic. Certainly there are models w/ critical pts of f where periodicity is impossible. Yet, over regular part, this is basically automatic.

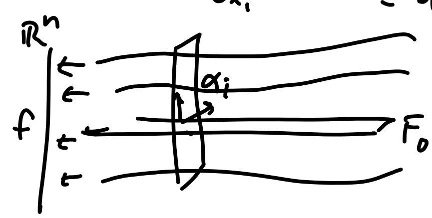
<u>Action-angle coordinates</u>: $f = (f_1, \dots f_n)$ loc. integrable system, near a regular value b_0.
Assume $F_0 = f^{-1}(b_0)$ compact closed (eg. M compact or f proper), by above it's Lagrangian.
Along F_0, $(df_1 \dots df_n)$ define basis of $N^*F_0 = T^*M/T^*F_0 \underset{\omega}{\cong} TF_0$.

$$df_i \longleftrightarrow X_i.$$

Working in a Weinstein nbd. we can identify locally M with T^*F_0, F_0 = zero section, nearby $f^{-1}(b)$ = graph.
Then $\exists$ <u>closed</u> 1-forms $\alpha_1, \dots \alpha_n$ on F_0 st.

$$f^{-1}(b_0 + (\lambda_1, \dots \lambda_n)) = \text{graph}\left(\sum \lambda_i \alpha_i + O(\lambda^2)\right)$$

$\leftarrow \alpha_i$ = lift of $\frac{\partial}{\partial x_i}$ to $T^*F_0 \underset{\cong}{\xrightarrow{d\pi}} T_{b_0}B$



pointwise $\alpha_1 .. \alpha_n$ basis of $\underset{\text{fiber}}{T^*F_0} \subset TM_{|F_0}$

and $\underset{T^*M}{df_i}(\underset{TM}{\alpha_j}) = \omega_0(\underset{TM}{\alpha_j}, \underset{TF_0}{X_i}) = \underset{T^*F_0}{\alpha_j}(\underset{TF_0}{X_i}) = \delta_{ij}$.

ie. $\{\alpha_j\}$ is dual basis to $\{X_i\}$.

Now: α_j closed $\Rightarrow$ on universal cover $\tilde{F}_0$, lift of α_j is exact $= d\theta_j$.
and flow of X_i lifts with $d\theta_j(X_i) = \delta_{ij}$, ie. $\theta_j : \tilde{F}_0 \to \mathbb{R}^n$, $X_j = \frac{\partial}{\partial \theta_j}$.
Since X_j pointwise linearly indept, $(\theta_1 .. \theta_n) : \tilde{F}_0 \to \mathbb{R}^n$ is a local diffeo hence a covering map hence a global diffeo. Moreover deck transformations of $\tilde{F}_0 \to F_0$ act by translations (<u>periods</u> of the 1-forms α_j) $\Rightarrow F_0 = \mathbb{R}^n/\text{lattice} \simeq T^n$.
(<u>Arnold-Liouville thm</u>: ‖ the compact reg. fibers of a c.i-system are tori).

* If we pick our coords. $(f_1, \dots f_n)$ on the base to be the affine ones, ie. $[\alpha_1] \dots [\alpha_n]$ is a basis of $H^1(F_0, \underline{\underline{\mathbb{Z}}})$ then the lattice of periods is $\mathbb{Z}^n \subset \mathbb{R}^n$, ie. we've identified θ_j = usual coordinates on torus $F_0 = T^n = (\mathbb{R}/\mathbb{Z})^n$ and $X_j = \partial/\partial\theta_j$. Thus: Ham. flows are <u>periodic</u>! Locally ~ Ham. T^n-action!

* The angle coords θ_j on F_0 (& any other regular fiber) are only defined up to an additive constant. Normalize these so that $\theta = 0$ is locally a Lagrangian section of f. (eg. some particular fiber of T^*F_0 in local model) (then by action of $X_i = \frac{\partial}{\partial \theta_i}$, so are all the other sections θ = const.). Then $\omega = \sum \omega_{ij}\, df_i \wedge d\theta_j$, and by constr. $X_{f_i} = \frac{\partial}{\partial \theta_i}$
$\Rightarrow \omega_{ij} = \delta_{ij}$, $\omega = \sum df_i \wedge d\theta_i$.

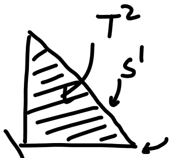
__Hence:__ near F_0, $\pi : M \to B$ looks like $(T^*B / T^*B^{\mathbb{Z}},\ \omega_0 = \Sigma\, df_i \wedge d\theta_i)$

affine coords $(f_1 \ldots f_n)$, $\mathrm{span}_{\mathbb{Z}}\left(\frac{\partial}{\partial f_i}\right) = TB^{\mathbb{Z}}$

dual coords. $(\theta_1 \ldots \theta_n)$ on T^*B, mod integer lattice $T^*B^{\mathbb{Z}}$

(Globally there are obstructions to the existence of Lagr. sections).

The local picture is boring but, as in the case of toric manifolds, it is the critical points of π that make things interesting.

__Ex:__ ⋆ toric manifolds, μ = moment map. Eg: $(\mathbb{CP}^2, \omega_0)$, $B \simeq$  fibers $= T^2$-orbits

$$\mu = \left(\frac{|x|^2}{|x|^2+|y|^2+|z|^2},\ \frac{|y|^2}{|x|^2+|y|^2+|z|^2}\right)$$

⋆ $(\mathbb{CP}^2, \omega_0)$, $f = \left(\frac{|x|^2 - |y|^2}{|x|^2+|y|^2+|z|^2},\ \frac{|xy - cz^2|}{|x|^2+|y|^2+|z|^2}\right)$

$= \mu_{S^1}$ moment map for S^1-action rotating x & y in opp dir^{ns}

(not normalized to affine str. coord.)

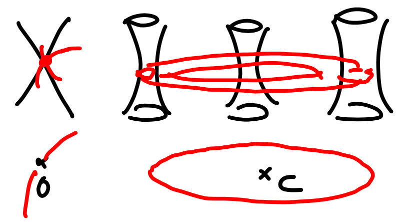
in reduced space $\mu_{S^1}^{-1}(\lambda)/S^1 \simeq \mathbb{CP}^1$ $[xy : z^2]$

levels are a family of curves "centered at" $\frac{xy}{z^2} = c$

The normalized coord. = amount of area enclosed.

(total area at level λ is $\frac{1-|\lambda|}{2}$).

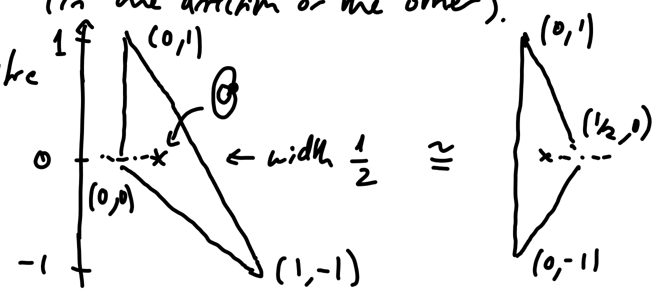
Analogue in $\mathbb{C}^2$ is $(|x|^2 - |y|^2,\ |xy - c|)$

already seen: $(x,y) \mapsto xy$ 

Only singularity is at origin, $f = (0, |c|)$. The sing. fiber $\simeq$ 

"Whitney sphere" = immersed Lagr. S^2 with one double point.

The nearby smooth fibers $f^{-1}(0,r)$ for $r \neq |c|$, are product-type resp Chekanov-type Lagr. tori; for r close to $|c|$ they are Ham. iso. to the outcome of Polterovich surgery at the double point of the sing. fiber (in one direction or the other).

The affine structure on the base B looks like 

at the branch cut, monodromy $\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

reflecting monodromy on H_1(fiber) (shear / S^1 orbit dir^n)

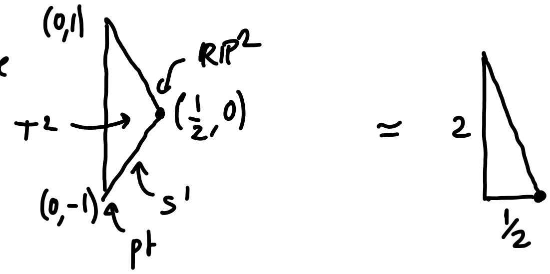
$\to$ For $c=0$ this reverts to the toric picture (fibers = product tori when $c=0$) ④

$\to$ For $c=\frac{1}{2}$ this becomes another interesting $\int$ system on $\mathbb{CP}^2$:

triangle ineq. $\Rightarrow$ $\dfrac{|xy-\frac{1}{2}z^2|}{|x|^2+|y|^2+|z|^2} \leq \dfrac{1}{2}$; equality case: normalize so $z \in \mathbb{R}$, then equality iff $y=-\bar{x}$.

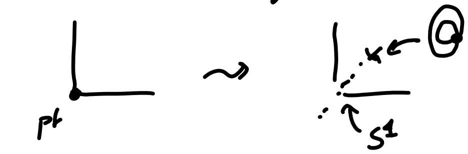
ie. for $c=\frac{1}{2}$, $f^{-1}(0,\frac{1}{2}) \simeq \mathbb{RP}^2$ (fixed pt set of cx. conjugation $(x:y:z)\mapsto(-\bar{y}:-\bar{x}:\bar{z})$).

The base looks like

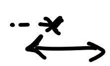


There is a calculus of such deformations of integrable systems in dim. 2 (Symington).

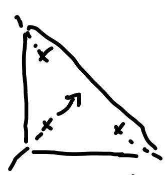
"Nodal trade" at $c=0$:



"Nodal slide" as c varies

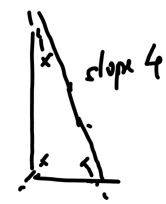


Starting from toric $\mathbb{CP}^2$, get



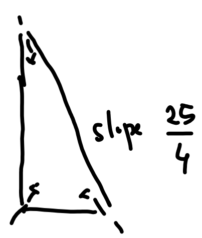
(1,1,1)

then successive nodal slides give (+ redraw cuts)



(1,1,4)

then

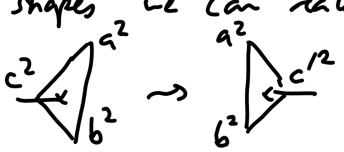


, …

(1,4,25) $\leftarrow$ classify triangular shapes by: det of primitive integer vectors along edges at each vertex

The set of shapes we can reach is all (a^2,b^2,c^2) st. $a^2+b^2+c^2=3abc$ ("Markov triples")

Nodal slide



with $c'=3ab-c$; "mutation"

‖ The fiber at a well-chosen point (affine barycenter) of the base is a monotone Lagrangian torus $T_{a^2,b^2,c^2} \subset \mathbb{CP}^2$. These are all different. (R. Vianna 2014)
(not related by Ham. iso or symplectomorphism)

$T_{1,1,1}$ = monotone product torus bounds 3 families of min. area holom. discs

$T_{1,1,4}$ = Chekanov 4 families (total multiplicity 5)

$T_{1,4,25}$ Vianna 10 41